Ministry of Higher Education and Scientific Research Northern Technical University
Oil & Gas. Tech. Eng. College/ Kirkuk

وزارة التعليم العالى العلمي والبحث العلمي
الجامعه التقنيه الشماليه
الكلية الهندسيه التقنية النفط و الغاز

Educational Bag

Scientific Department: Fuel and Energy Techniques Engineering

Course Name: الميكانيك الهندسي

Semester: الاول

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Course Outline:

❖ Part-I- Statics

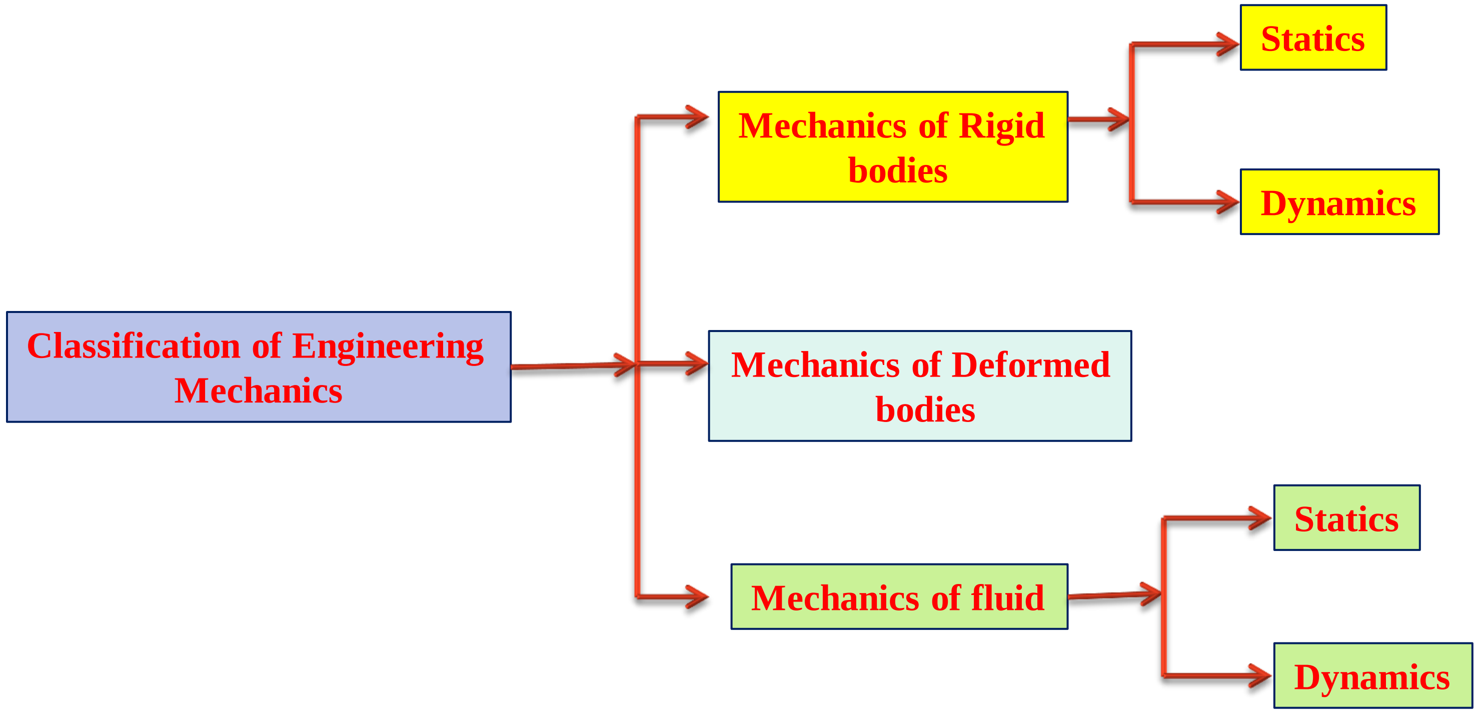
- Principles of Statics.
- Resultants of Force Systems.
- Equilibrium of Force Systems.
- Friction.
- Centers of Gravity.
- Moment of Inertia.

❖ Part- II- Dynamics

- Principles of Dynamics.
- Rectilinear Motion.
- Curvilinear Motion.
- Rotation.
- Energy and Work.
- Mechanical Vibration

Basic Concepts

❖ **Mechanics:** is a branch of the physical sciences that is concerned with the state of rest or motion of bodies that are subjected to the action of forces.



- **Statics :** deals with the equilibrium of bodies, that is, those that are either at rest or move with a constant velocity.
- **Dynamics:** is concerned with the accelerated motion of bodies.

❑ **Fundamental Concepts :** Before we begin our study of engineering mechanics, it is important to understand the meaning of certain fundamental concepts and principles.

- **Basic Quantities.** The following four quantities are used throughout mechanics.

  - **Length.** Length is used to locate the position of a point in space and thereby describe the size of a physical system. Some important length conversions factors as shown below.

1 kilometer = 1000 meters	1 mile = 1760 yards
1 meter = 100 centimeters	1 mile = 5280 feet
1 centimeter = 10 millimeters	1 yard = 3 feet
	1 foot = 12 inches

  - **Time.** Time is conceived as a succession of events. Although the principles of statics are time-independent, this quantity plays an important role in the study of dynamics. Some important time conversions factors as shown below.

1 hour = 60 minutes	1 week = 7 days
1 minute = 60 seconds	1 year = 365 days
1 hour = 60 minutes	1 year = 12 months
1 day = 24 hours	1year = 52 weeks

- **Mass.** Mass is a measure of a quantity of matter that is used to compare the action of one body with that of another. This property manifests itself as a gravitational attraction between two bodies and provides a measure of the resistance of matter to a change in velocity. Some important mass conversion factor as shown below

$$1 \text{ Kg} = 2.204 \text{ lb}_m$$

- **Force.** In general, force is considered as a "push" or "pull" exerted by one body on another. This interaction can occur when there is direct contact between the bodies, such as a person pushing on a wall, or it can occur through a distance when the bodies are physically separated. Some important force units and conversion factors as shown below.

	newton (SI unit)	dyne	kilogram-force, kilopond	pound-force	poundal
1 N	$\equiv 1$ kg·m/s²	$= 10^5$ dyn	≈ 0.10197 kp	≈ 0.22481 lb_f	≈ 7.2330 pdl
1 dyn	$= 10^{-5}$ N	$\equiv 1$ g·cm/s²	$\approx 1.0197\times10^{-6}$ kp	$\approx 2.2481\times10^{-6}$ lb_f	$\approx 7.2330\times10^{-5}$ pdl
1 kp	$= 9.80665$ N	$= 980665$ dyn	$\equiv g_n\cdot(1$ kg$)$	≈ 2.2046 lb_f	≈ 70.932 pdl
1 lb_f	≈ 4.448222 N	≈ 444822 dyn	≈ 0.45359 kp	$\equiv g_n\cdot(1$ lb$)$	≈ 32.174 pdl
1 pdl	≈ 0.138255 N	≈ 13825 dyn	≈ 0.014098 kp	≈ 0.031081 lb_f	$\equiv 1$ lb·ft/s²

- **Idealizations.** idealizations or models are used in mechanics in order to simplify the application of the theory. The following are three important idealizations.

- **Particle.** A particle has a mass, but a size that can be neglected. For example, the size of the earth is insignificant compared to the size of its orbit, and therefore the earth can be modeled as a particle when studying its orbital motion.

Fig.1, Three forces act on the ring. Since these forces all meet at point, then for any force analysis, we can assume the ring to be represented as a particle.

- **Rigid Body.** A rigid body can be considered as a combination of a large number of particles in which all the particles remain at a fixed distance from one another, both before and after applying a load.

  - ✓ This model is important because the body's shape does not change when a load is applied, and so we do not have to consider the type of material from which the body is made.

  - ✓ In most cases, the actual deformations occurring in structures, machines, mechanisms, and the like are relatively small, and the rigid-body assumption is suitable for analysis.

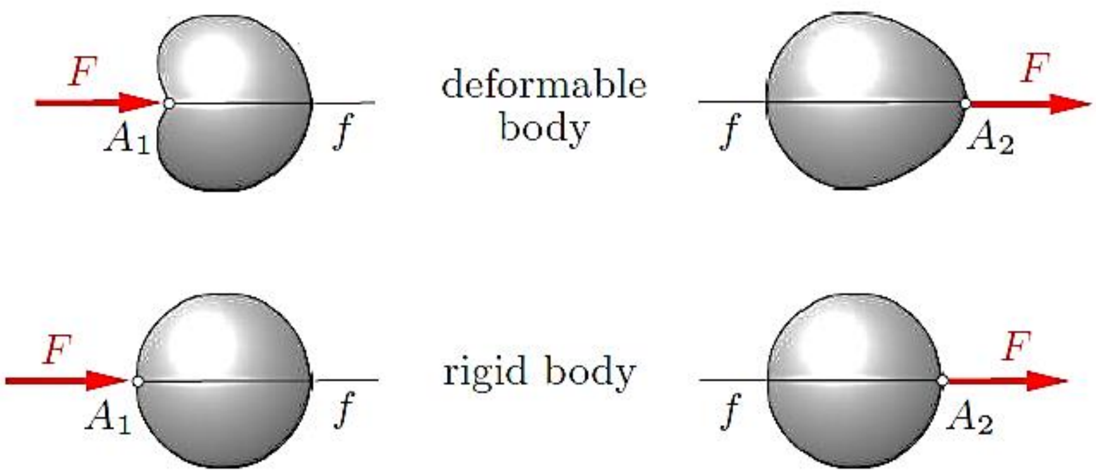


Fig.,2

- **Concentrated Force.** A concentrated force represents the effect of a loading which is assumed to act at a point on a body.

  ✓ We can represent a load by a concentrated force, provided the area over which the load is applied is very small compared to the overall size of the body.

  ✓ An example would be the contact force between a wheel and the ground.

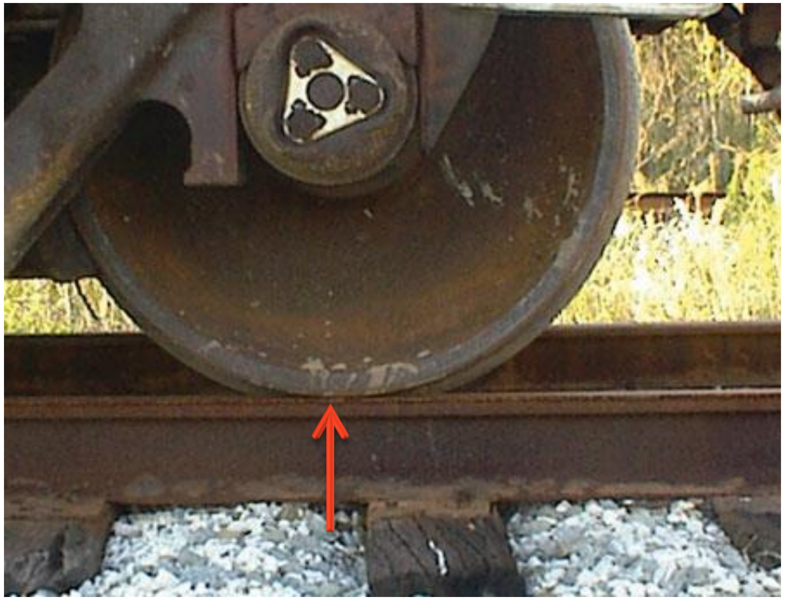


Fig.3, Steel is a common engineering material that does not deform very much under load. Therefore, we can consider this railroad wheel to be a rigid body acted upon by the concentrated force of the rail.

- **Newton's Three Laws of Motion.** Engineering mechanics is formulated on the basis of Newton's three laws of motion, the validity of which is based on experimental observation. These laws apply to the motion of a particle as measured from a *nonaccelerating* reference frame. They may be briefly stated as follows.

  - **First Law.** A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is not subjected to an unbalanced force, Fig.4a, .

  - **Second Law.** A particle acted upon by an unbalanced force **F** experiences an acceleration a that has the same direction as the force and a magnitude that is directly proportional to the force, Fig.4b. If **F** is applied to a particle of mass **m** , this law may be expressed mathematically as

$$F = ma \quad \ldots\ldots\ldots\ldots\ldots(1)$$

  - **Third Law.** The mutual forces of action and reaction between two particles are equal, opposite, and collinear, Fig. 4c,.

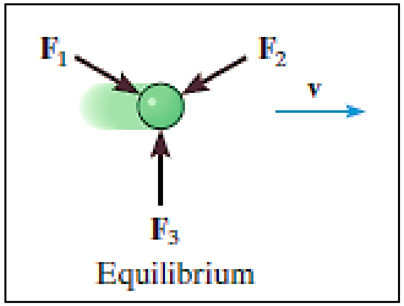


Fig.4a,

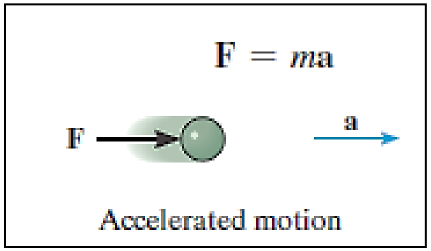


Fig.4b,

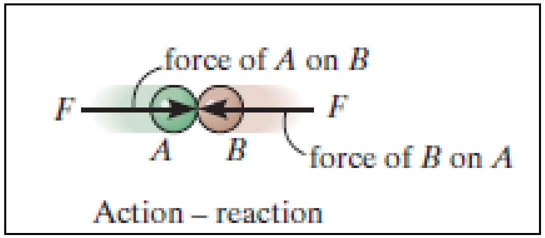
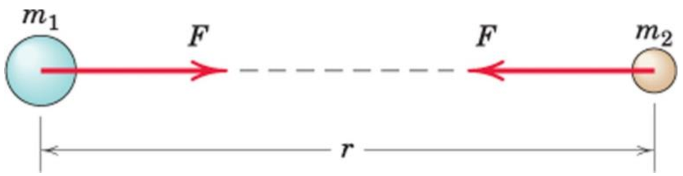


Fig.4c,

- **Newton's Law of Gravitational Attraction.** Shortly after formulating his three laws of motion, Newton postulated a law governing the gravitational attraction between any two particles. Stated mathematically,

$$F = G\frac{m_1 m_2}{r^2} \quad \text{.....................(2)}$$



where
F = force of gravitation between the two particles
G = universal constant of gravitation; according to experimental evidence, $G = 66.73(10^{-12})\ m^3/(\text{kg} \cdot s^2)$
m_1, m_2 = mass of each of the two particles
r = distance between the two particles

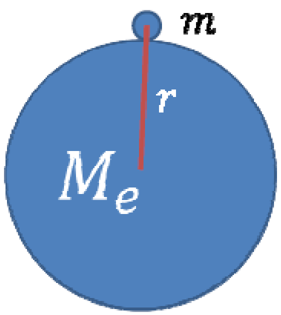
- **Gravitational Attraction of the Earth**

✓ **Weight of a Body**: If a particle is located at or near the surface of the earth, the only significant gravitational force is that between the earth and the particle.
- Weight of a particle having mass $\boldsymbol{m_1 = m}$:
- Assuming earth to be a nonrotating sphere of constant density and having mass $m_2 = M_e$
- r = distance between the earth's center and the particle
- g= acceleration due to gravity =9.81 m/s^2

$$W = G\frac{mM_e}{r^2} \quad \text{..........(3)}$$

$$g = \frac{GM_e}{r^2} \quad \text{..............(4)}$$

$$W = m\,g$$



Systems of units

In engineering mechanics length, mass, time and force are the basic units used therefore; the following are the units systems are adopted in the engineering mechanics

- **International System of Units (SI):**

  In SI system of units the basic units are length, time, and mass which are arbitrarily defined as the meter (m), second (s), and kilogram (kg). Force is the derived unit.

$$\mathbf{1N = 1\ kg.\ m/}\boldsymbol{s^2}$$

- **CGS systems of units**

  In CGS system of units, the basic units are length, time, and mass which are arbitrarily defined as the centimeter (cm), second (s), and gram (g). Force is the derived units

$$\mathbf{1\ Dyne = 1\ g.\ \ cm/}\boldsymbol{s^2}$$

- **British systems of units**

  In CGS system of units, the basic units are length, time, and mass which are arbitrarily defined as the foot , second, and pound. Force is the derived units

- **U.S. Customary Units**

  The basic units are length, time, and force which are arbitrarily defined as the foot (ft), second (s), and pound (lb). Mass is the derived unit, Ex.1. Convert 2 km/h to m/s How many ft/s is this?

Sol.

Since 1 km = 1000 m and 1 h = 3600 s , the factors of conversion are arranged in the following order, so that a cancellation of the units can be applied:

$$2\ \mathrm{km/h} = \frac{2\ \cancel{\mathrm{km}}}{\cancel{\mathrm{h}}}\left(\frac{1000\ \mathrm{m}}{\cancel{\mathrm{km}}}\right)\left(\frac{1\ \cancel{\mathrm{h}}}{3600\ \mathrm{s}}\right)$$

$$= \frac{2000\ \mathrm{m}}{3600\ \mathrm{s}} = 0.556\ \mathrm{m/s}$$

1 ft = 0.3048 m. Thus,

$$0.556\ \mathrm{m/s} = \left(\frac{0.556\ \cancel{\mathrm{m}}}{\mathrm{s}}\right)\left(\frac{1\ \mathrm{ft}}{0.3048\ \cancel{\mathrm{m}}}\right)$$

$$= 1.82\ \mathrm{ft/s}$$

Ex.2. Convert the quantities 300 lb .s and 52 slug/ft^3 to appropriate SI units.

Sol.

1 lb = 4.448 N , 1 slug = 14.59 kg and 1 ft = 0.3048 m , then

1) $$300 \text{ lb}\cdot\text{s} = 300\ \cancel{\text{lb}}\cdot\text{s}\left(\frac{4.448 \text{ N}}{1\ \cancel{\text{lb}}}\right)$$
$$= 1334.5 \text{ N}\cdot\text{s} = 1.33 \text{ kN}\cdot\text{s}$$

2) $$52 \text{ slug/ft}^3 = \frac{52\ \cancel{\text{slug}}}{\cancel{\text{ft}^3}}\left(\frac{14.59 \text{ kg}}{1\ \cancel{\text{slug}}}\right)\left(\frac{1\ \cancel{\text{ft}}}{0.3048 \text{ m}}\right)^3$$
$$= 26.8(10^3) \text{ kg/m}^3$$
$$= 26.8 \text{ Mg/m}^3$$

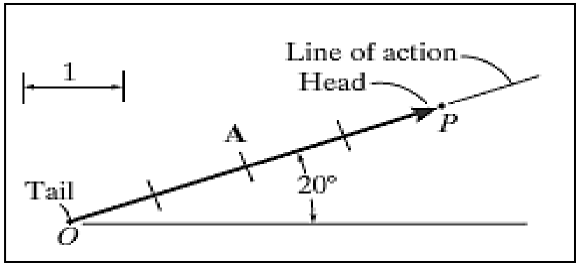
○ Scalars and Vectors

✓ **Scalar.** A *scalar* is any positive or negative physical quantity that can be completely specified by its ***magnitude*** . Examples of scalar quantities include,

- length
- mass
- time

✓ **Vector.** A *vector* is any physical quantity that requires both a ***magnitude*** and a ***direction*** for its complete description. Examples of vectors encountered in statics are,

- force,
- Position
- moment.

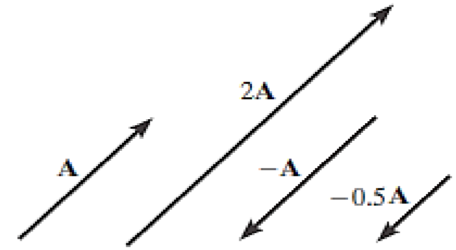


- A vector is shown graphically by an arrow.
- The length of the arrow represents the magnitude of the vector.
- the angle θ between the vector and a fixed axis defines the direction of its line of action.
- The head or tip of the arrow indicates the sense of direction of the vector.
- Vector quantities are represented by boldface letters such as **A** , and the magnitude of a vector is italicized, *A.*
- For handwritten work, it is often convenient to denote a vector quantity by simply drawing an arrow above it, $\vec{A}$.

❖ Vector Operations

○ Multiplication and Division of a Vector by a Scalar.

- ✓ If a vector is multiplied by a positive scalar, its magnitude is increased by that amount.
- ✓ Multiplying by a negative scalar will also change the directional sense of the vector.
- ✓ Graphic examples of these operations are shown in Fig. below.

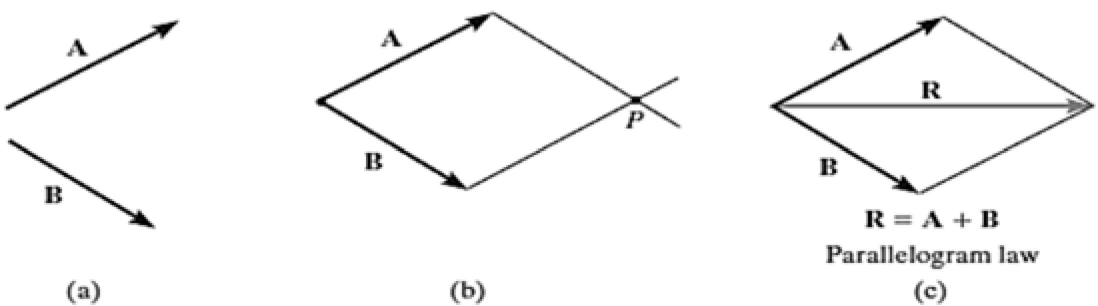


Scalar multiplication and division

○ Vector Addition.

- ✓ All vector quantities obey the **parallelogram law of addition** .
- ✓ To illustrate, the two " component " vectors **A** and **B** in (Fig.a) are added to form a " resultant " vector **R = A + B** using the following procedure:

1. First join the tails of the components at a point to make them concurrent, Fig.b .
2. From the head of **B** , draw a line parallel to **A** .
3. Draw another line from the head of **A** that is parallel to **B** .
4. These two lines intersect at point P to form the adjacent sides of a parallelogram.
5. The diagonal of this parallelogram that extends to P forms **R** , which then represents the resultant vector $\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B}$ Fig.c,



- ✓ We can also add **B** to **A** , Fig.2*a* , using the *triangle rule* , which is a special case of the parallelogram law.

1. whereby vector **B** is added to vector **A** in a “head-to-tail” fashion, i.e., by connecting the head of **A** to the tail of **B** , Fig. 2*b* .
2. The resultant **R** extends from the tail of **A** to the head of **B** . In a similar manner, **R** can also be obtained by adding **A** to **B** , Fig. 2c.
3. By comparison, it is seen that vector addition is commutative; in other words, the vectors can be added in either order, i.e.,

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} .$$

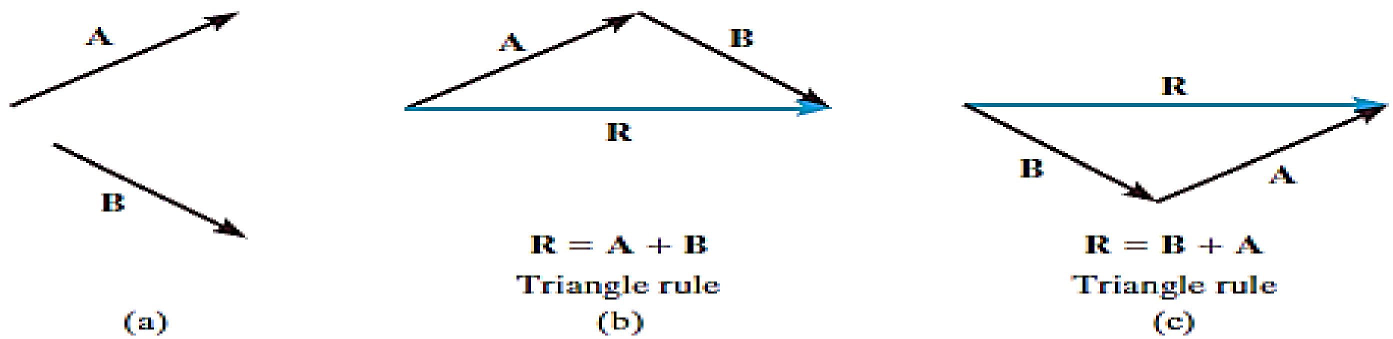


Fig.2

- ✓ As a special case, if the two vectors **A** and **B** are *collinear* , i.e., both have the same line of action, the parallelogram law reduces to an *algebraic* or *scalar addition* $R = A + B$, as shown in Fig.3.,

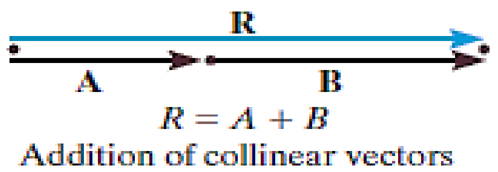


Fig.3

○ **Vector Subtraction.** The resultant of the *difference* between two vectors **A** and **B** of the same type may be expressed as

$$\boldsymbol{R'} = \boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$$

✓ Subtraction is therefore defined as a special case of addition, so the rules of vector addition also apply to vector subtraction as shown in Fig.4,

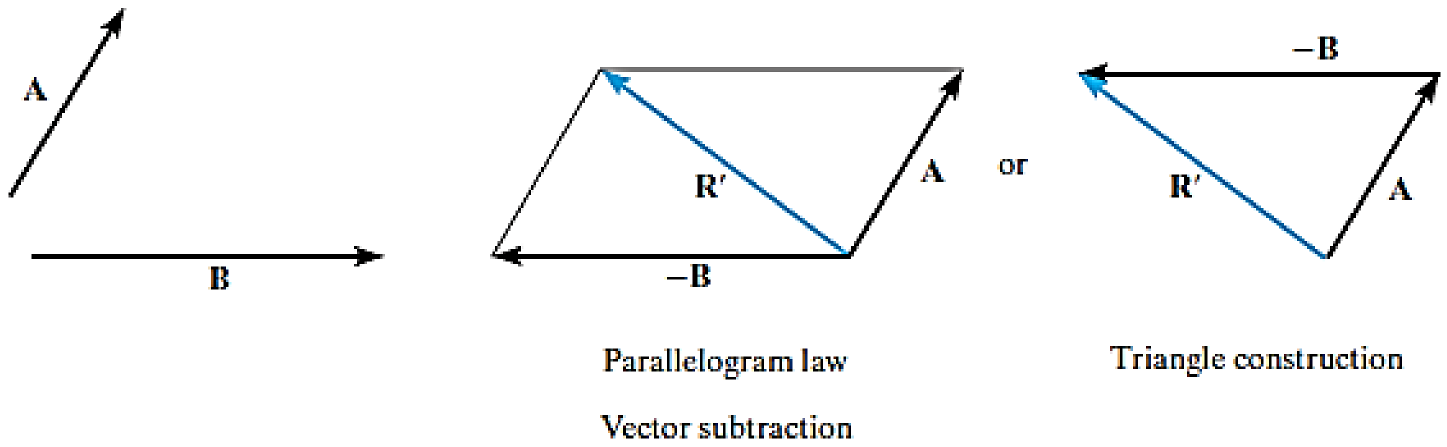
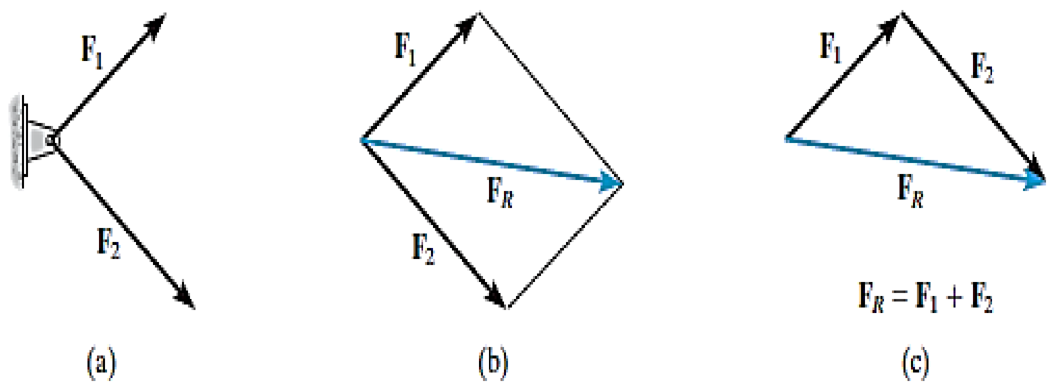


Fig.4

❖ Vector Addition of Forces

- ### Finding a Resultant Force.

The two component forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the pin in Fig.5*a* can be added together to form the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$, as shown in Fig. 5*b*. From this construction, or using the triangle rule, Fig.5*c*, we can apply the law of cosines or the law of sines to the triangle in order to obtain the magnitude of the resultant force and its direction.



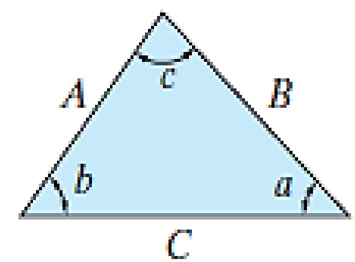
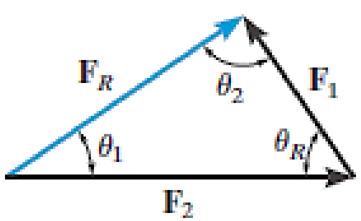


Fig.5



$$F_R = \sqrt{F_1^2 + F_2^2 - 2\,F_1F_2\cos\theta_R}$$

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_R}{\sin\theta_R}$$

Or

Cosine law:

$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

Sine law:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

- **Finding the Components of a Force.** Sometimes it is necessary to resolve a force into two *components* in order to study its pulling or pushing effect in two specific directions.

- In Fig. 6 a , **F** is to be resolved into two components along the two members, defined by the u and v axes.

- To determine the magnitude of each component for Fig.6a, a parallelogram is constructed.
  - ✓drawing lines starting from the tip of **F** , one line parallel to u , and the other line parallel to v.
  - ✓ These lines then intersect with the v and u axes, forming a parallelogram.
  - ✓ The force components $\mathbf{F}u$ and $\mathbf{F}v$ are then established by simply joining the tail of **F** to the intersection points on the u and v axes, Fig.6 b.
  - ✓ This parallelogram can then be reduced to a triangle, which represents the triangle rule, Fig. 6 c .
  - ✓ From this, the law of sines can then be applied to determine the unknown magnitudes of the components.

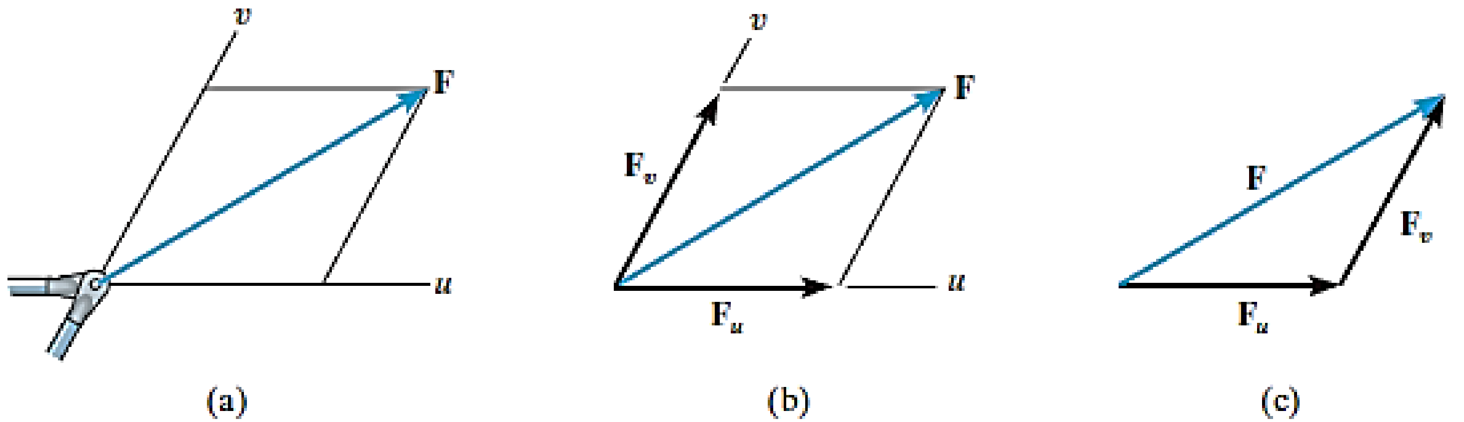
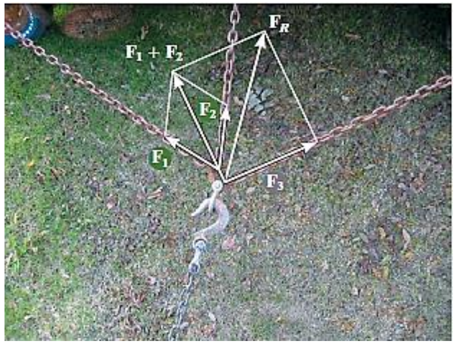
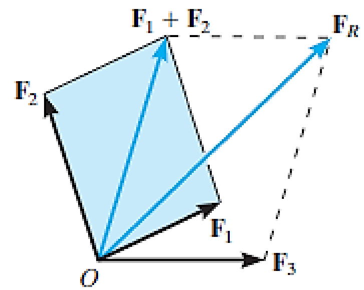


Fig.6,

- **Addition of Several Forces.** If more than two forces are to be added, successive applications of the parallelogram law can be carried out in order to obtain the resultant force. For example, if three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act at a point O, as show in below Figs7.,

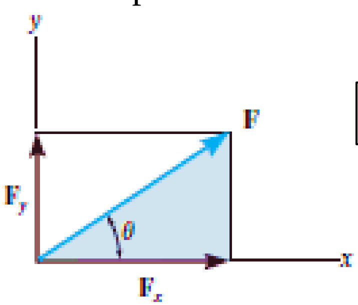




Figs7.

Fig.8a

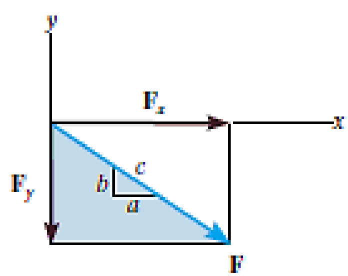
- ✓ Resultant of any two of the forces = $\boldsymbol{F}_1 + \boldsymbol{F}_2$
- ✓ Resultant of all three forces, i.e $\boldsymbol{F}_R = (\boldsymbol{F}_1 + \boldsymbol{F}_2) + \boldsymbol{F}_3$.
- ✓ To determine the numerical values for the magnitude and direction of the resultant rectangular-component method as show below,



$$\boldsymbol{F = F_x + F_y}$$

$$\boldsymbol{F_x = F\cos\theta}$$

$$\boldsymbol{F_y = F\sin\theta}$$



$$\frac{F_x}{F} = \frac{a}{c} \rightarrow F_x = F\left(\frac{a}{c}\right)$$

$$\frac{F_y}{F} = \frac{b}{c} \rightarrow F_y = -F\left(\frac{b}{c}\right)$$

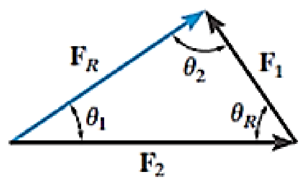
- The direction of **F** can also be defined using a small "slope" triangle,
- The **y** component is a negative scalar since **Fy** is directed along the negative y axis.

Ex.3. The screw eye in Fig.8a is subjected to two forces, **F**1 and **F**2. Determine the magnitude and direction of the resultant force.

Sol.

- **Parallelogram Law.** The parallelogram is formed by drawing a line from the head of $\boldsymbol{F}_1$ that is parallel to $\boldsymbol{F}_2$, and another line from the head of $\boldsymbol{F}_2$ that is parallel to $\boldsymbol{F}_1$. The resultant force $\boldsymbol{F}_R$ extends to where these lines intersect at point A , Fig. 8 b . The two unknowns are the magnitude of $\mathbf{F}_R$ and the angle θ (theta).

- **Trigonometry.** From the parallelogram, the vector triangle is constructed, Fig. 8 c . Using the law of cosines



$$F_R = \sqrt{F_1^2 + F_2^2 - 2\,F_1 F_2 \cos\theta_R}$$

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_R}{\sin\theta_R}$$

$$\boldsymbol{F_R = \sqrt{(100\,N)^2 + (150\,N)^2 \quad - 2(100\,N)(150N)\cos 115^\circ} = 212.6N \approx 213N}$$

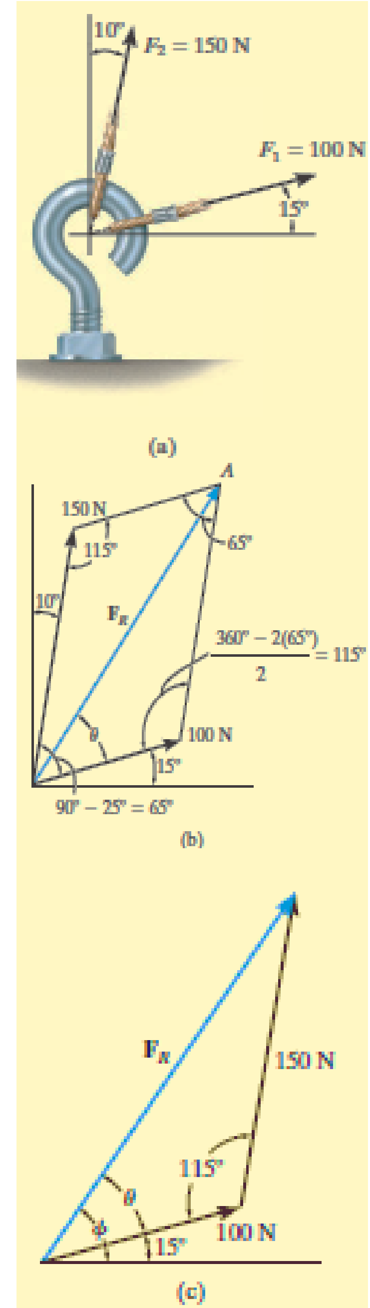
➢ Applying the law of sines to determine θ,

$$\boldsymbol{\frac{150N}{\sin\theta} = \frac{212.6N}{\sin 115^\circ}}\,, \qquad \boldsymbol{\sin\theta = \frac{150N}{212.6N}\,(\sin 115^\circ) = \theta = 39.8^\circ}$$

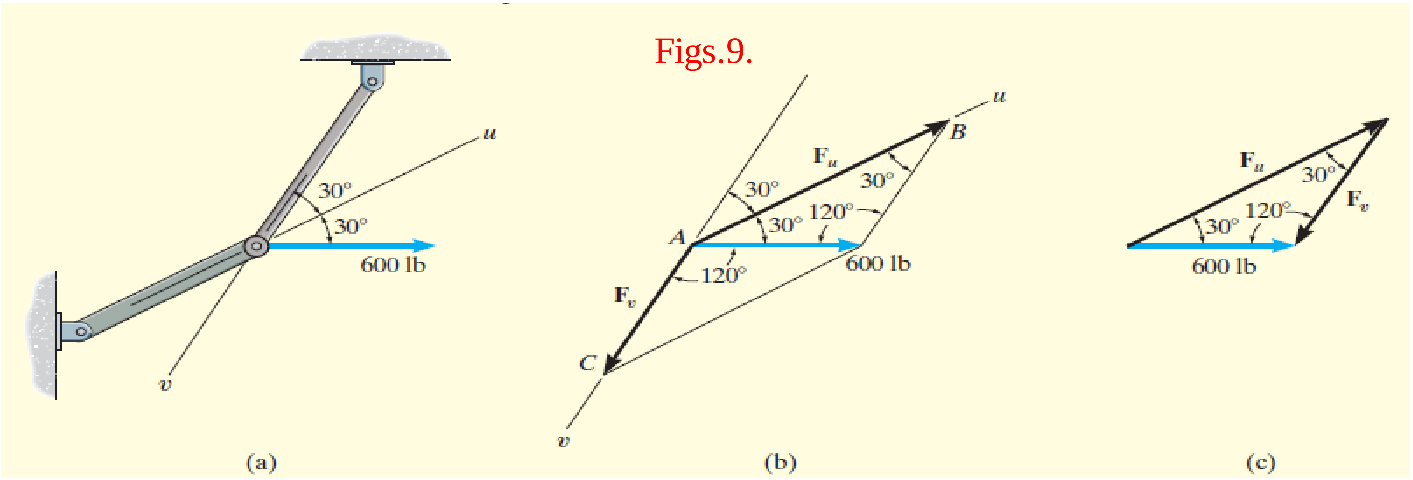
➢ Thus, the direction $\emptyset$ (phi) of $\mathbf{F}_R$, measured from the horizontal, is

$$\boldsymbol{\emptyset = 39.8^\circ + 15^\circ = 54.8^\circ}$$

NOTE: The results seem reasonable since Fig. 8 b shows $\boldsymbol{F}_R$ to have a magnitude larger than its components and a direction that is between them.

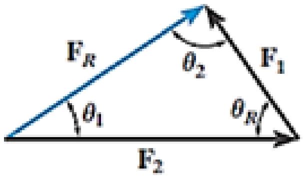


Ex.4. Resolve the horizontal 600-lb force in Fig. 9a into components acting along the u and v axes and determine the magnitudes of these components.



Sol.

- The parallelogram is constructed by extending a line from the head of the 600-lb force parallel to the v axis until it intersects the u axis at point **B** , Fig.9b.
- The arrow from **A** to **B** represents $\boldsymbol{F_u}$.
- Similarly, the line extended from the head of the 600-lb force drawn parallel to the $u\ axis$ intersects the $v\ axis$ at point **C**, which gives $\boldsymbol{F_v}$.
- The vector addition using the triangle rule is shown in Fig.9c . The two unknowns are the magnitudes of $\boldsymbol{F_u}$ and $\boldsymbol{F_v}$. Applying the law of sines,



$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_R}{\sin\theta_R}$$

$$\frac{\boldsymbol{F_u}}{\sin 120^\circ} = \frac{600\ Ib}{\sin 30^\circ}, \quad \boldsymbol{F_u = 1039\ lb} \qquad \frac{\boldsymbol{F_v}}{\sin 30^\circ} = \frac{600\ Ib}{\sin 30^\circ}, \quad \boldsymbol{F_v = 600\ lb}$$

NOTE: The result for *Fu* shows that sometimes a component can have a greater magnitude than the resultant.

Ex 5. It is required that the resultant force acting on the eyebolt in **Fig.10*a*** be directed along the positive $x\ axis$ and that $\boldsymbol{F_2}$ have a *minimum* magnitude. Determine this magnitude, the angle θ, and the corresponding resultant force.

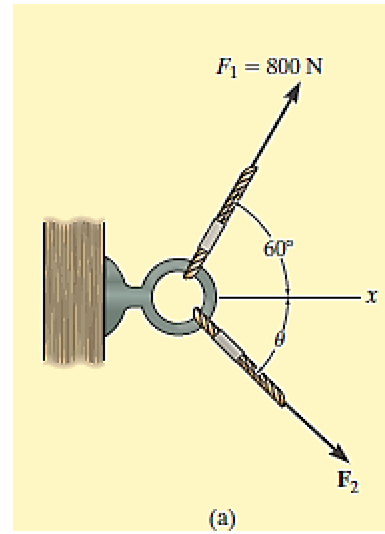
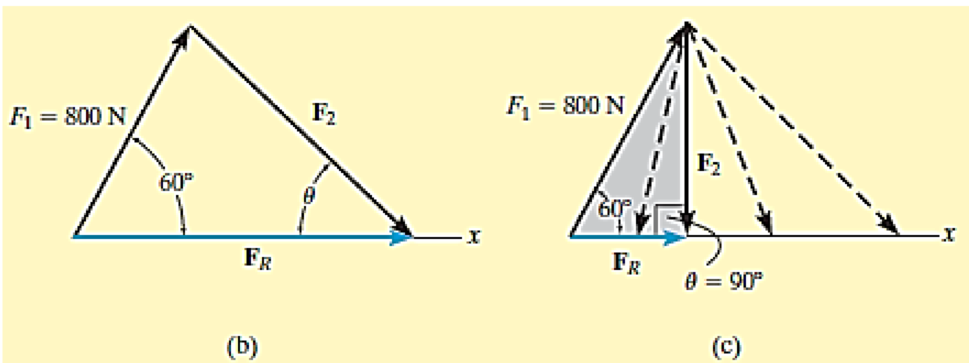


Fig.10*a*

Sol.



- The triangle rule for $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$ is shown in Fig. 10*b* .
- Since the magnitudes (lengths) of $\mathbf{F}_R$ and $\mathbf{F}_2$ are not specified, then $\mathbf{F}_2$ can actually be any vector that has its head touching the line of action of $\mathbf{F}_R$. Fig. 10*c.*
- The magnitude of $\mathbf{F}_2$ is a *minimum* or the shortest length when its line of action is *perpendicular* to the line of action of $\mathbf{F}_R$, that is, when

$$\boldsymbol{\theta = 90^{\circ}}$$

- Since the vector addition now forms the shaded right triangle, the two unknown magnitudes can be obtained by trigonometry.

$$\boldsymbol{F_R = (800\text{N})\cos 60^{\circ} = 400\ \text{N}}$$

$$\boldsymbol{F_2 = (800N)\sin 60^{\circ} = 693\ \text{N}}$$

❖ Rectangular Components: Two Dimensions.

- Vectors $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$ are rectangular components of $\boldsymbol{F}$.

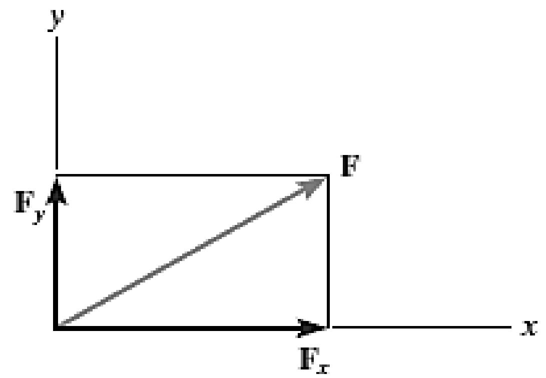
  - The resultant force is determined from the algebraic sum of its components.

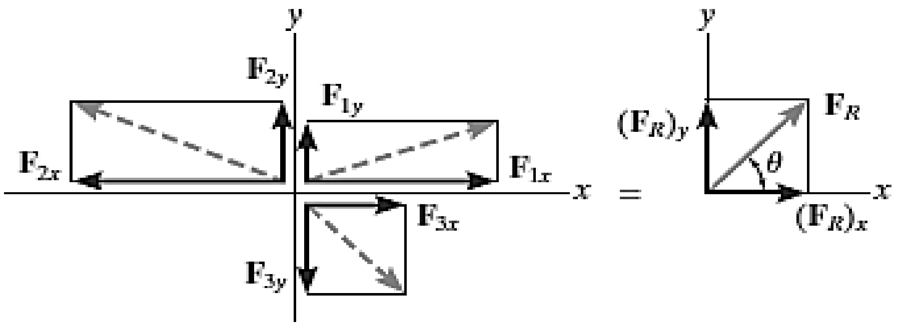
$$(F_R)_x = \Sigma F_x$$

$$(F_R)_y = \Sigma F_y$$

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2}$$

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|$$

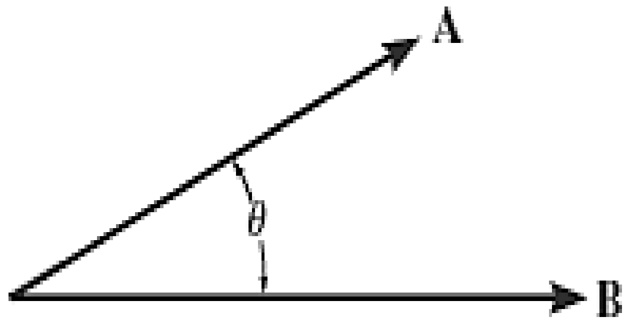




❖ Dot Product

- The dot product between two vectors **A** and **B** yields a scalar. If **A** and **B** are expressed in Cartesian vector form, then the dot product is the sum of the products of their ***x , y , and z components.***

$$\boxed{\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= AB\cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}}$$

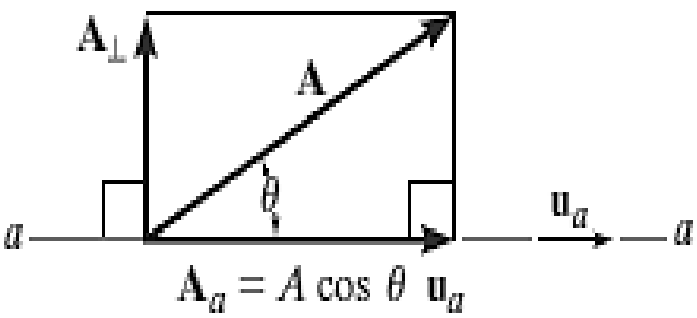


- The dot product can be used to determine the angle between **A** and **B** .

$$\boxed{\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)}$$

- The dot product is also used to determine the projected component of a vector **A** onto an *axis aa* defined by its unit vector u_a.

$$\boxed{\mathbf{A}_a = A\cos\theta\,\mathbf{u}_a = (\mathbf{A}\cdot\mathbf{u}_a)\mathbf{u}_a}$$



Ex.6. Determine the x and y components of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ acting on the boom shown in Fig.11*a* .

Sol.

Scalar Notation.

- By the parallelogram law, F_1 is resolved into x *and* y *components*, Fig. 11*b* . Since $\boldsymbol{F_{1x}}$ acts in the $-x$ direction, and $\boldsymbol{F_{1y}}$ acts in the $+y$ direction, we have

$$\boldsymbol{F_{1x} = -200\sin 30^\circ\ N = -100\ N = 100N\ \leftarrow}$$

$$\boldsymbol{F_{1y} = 200\ \cos 30^\circ\ N = 173N = 173\ N\ \uparrow}$$

- The force $\mathbf{F_2}$ is resolved into its x and y components, as shown in Fig. 11*c.*
- Obtain the angle u, e.g., $\boldsymbol{\theta = \tan^{-1}(\frac{5}{12})}$,
- Determine the magnitudes of the components in the same manner as for $\mathbf{F_1}$ using proportional parts of similar triangles, i.e.,

$$\frac{\boldsymbol{F_{2x}}}{\mathbf{260\ N}} = \frac{\mathbf{12}}{\mathbf{13}} \rightarrow \boldsymbol{F_{2x}} = \mathbf{260}\ \boldsymbol{N}(\frac{\mathbf{12}}{\mathbf{13}}) = \mathbf{240N\ and}\ \boldsymbol{F_{2y}} = \mathbf{260\ N}(\frac{\mathbf{5}}{\mathbf{13}}) = \mathbf{100\ N}$$

- Notice

➢ The magnitude of the horizontal component , $\mathbf{F_{2x}}$, was obtained by multiplying the force magnitude by the ratio of the horizontal leg of the slope triangle divided by the hypotenuse.

➢ The magnitude of the vertical component , $\mathbf{F_{2y}}$, was obtained by multiplying the force magnitude by the ratio of the vertical leg divided by the hypotenuse

∴using scalar notation to represent these components, we have

$\boldsymbol{F_{2x}} = \mathbf{240N} \rightarrow$ $\qquad\qquad$ $\boldsymbol{F_{2y}} = \boldsymbol{-100\ N} = \mathbf{100\ N}\downarrow$

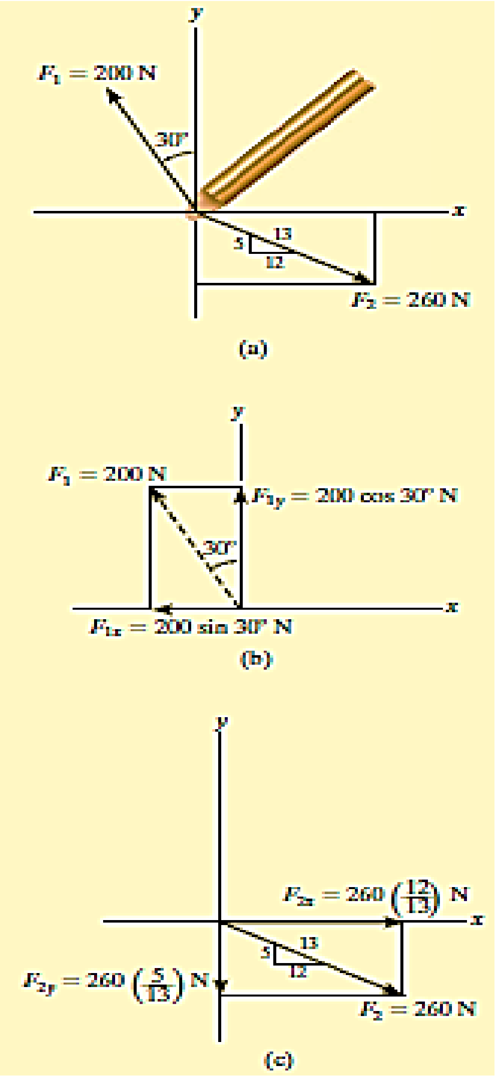


Fig.11

Ex.7.The link in Fig. 12*a* is subjected to two forces **F**1 and **F**2 . Determine the magnitude and direction of the resultant force.

Sol.I

Scalar Notation.

- First we resolve each force into its *x* and *y* components, Fig. 12 *b* , then we sum these components algebraically.

$$\overset{+}{\rightarrow} (\boldsymbol{F_R})_x = \sum \boldsymbol{F_x} \qquad (\boldsymbol{F_R})_x = \mathbf{600\cos 30^\circ} \boldsymbol{N} - \mathbf{400\sin 45^\circ}\ \boldsymbol{N} = \mathbf{236.8}\boldsymbol{N} \rightarrow$$

$$+\uparrow (\boldsymbol{F_R})_y = \sum \boldsymbol{F_y} \qquad (\boldsymbol{F_R})_y = \mathbf{600\sin 30^\circ} \boldsymbol{N} + \mathbf{400\cos 45^\circ}\ \boldsymbol{N} = \mathbf{582.8}\boldsymbol{N} \uparrow$$

- The resultant force, shown in Fig. 12 c , has a magnitude of

$$\boldsymbol{F_R} = \sqrt{(\mathbf{236.8}\ \boldsymbol{N})^2 + (\mathbf{582.8}\ \boldsymbol{N})^2} = \mathbf{629}\boldsymbol{N}$$

- From the vector addition,

$$\boldsymbol{\theta} = \tan^{-1}\left(\frac{582.8N}{236.8N}\right) = 67.9^\circ$$

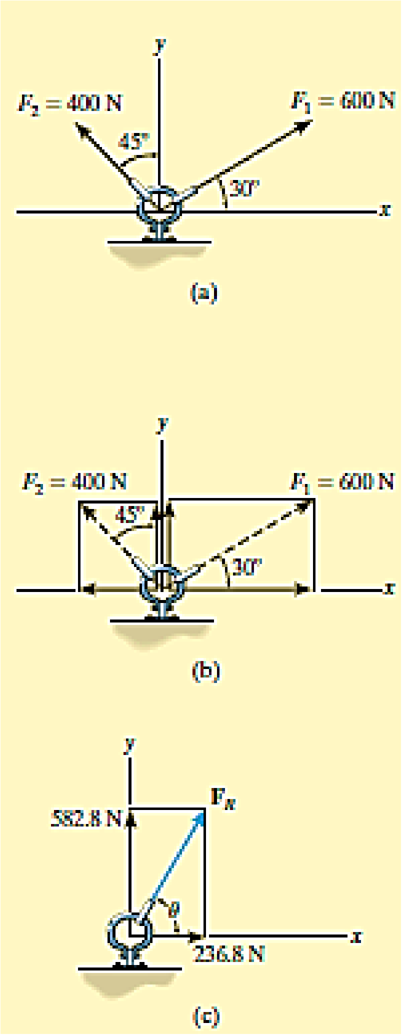


Fig. 12

Ex.8.The end of the boom O in Fig. 13 a is subjected to three concurrent and coplanar forces. Determine the magnitude and direction of the resultant force.

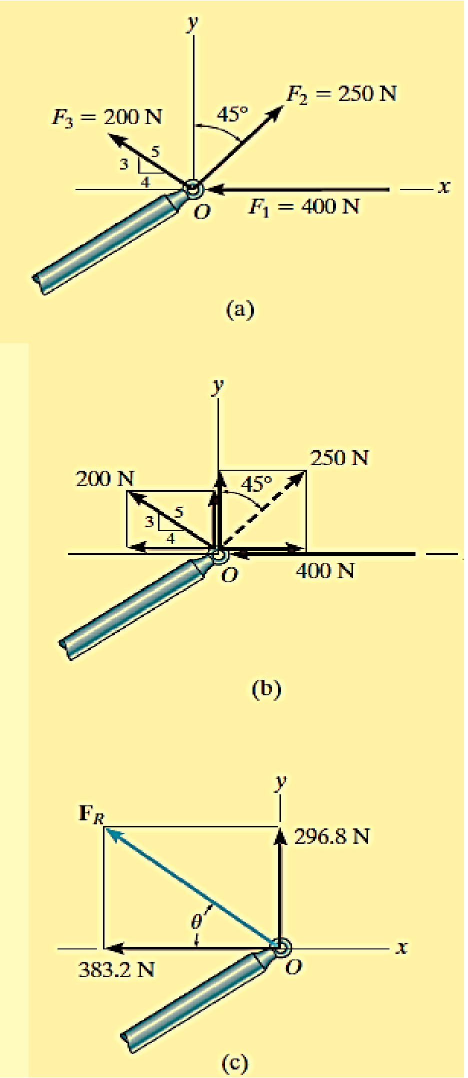


Fig. 13

Sol.

- Each force is resolved into its x and y components, Fig. 13b . Summing the x components, we have

$$\overset{+}{\rightarrow} (F_R)_x = \sum F_x \qquad (F_R)_x = -400\,N + 250 \sin 45^\circ\,N - 200\left(\frac{4}{5}\right)N = -383.2N = 383.2 \leftarrow$$

- The negative sign indicates that F_{Rx} acts to the left, i.e., in the negative x **direction**, as noted by the small arrow. Obviously, this occurs because F_1 and F_3 in Fig.13b contribute a greater pull to the left than F_2 which pulls to the right. Summing the y components yields

$$+\uparrow\ (F_R)_y = \sum F_y \qquad (F_R)_y = 250 \cos 45^\circ N + 200\left(\frac{3}{5}\right)N = 296.8N\ \uparrow$$

- The resultant force, shown in Fig.13 c , has a magnitude of

$$F_R = \sqrt{(-383.2\ N)^2 + (296.8\ N)^2} = 485N$$

- From the vector addition in Fig. 13c, the direction angle θ is

$$\theta = \tan^{-1}\left(\frac{296.8}{383.2}\right) = 37.8^\circ$$

NOTE: Application of this method is more convenient, compared to using two applications of the parallelogram law, first to add **F**1 and **F**2 then adding **F**3 to this resultant.

Force System Resultants

❖ Moment of a Force:

➢ What is the Moment?

- When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force.

- This tendency to rotate is sometimes called a torque , also called the moment of a force or simply the moment .

- Consider there is a wrench used to unscrew the bolt in Fig. 14 a .

- If a force is applied to the handle of the wrench it will tend to turn the bolt about point O (or the *z axis*).

- The magnitude of the moment is directly proportional to the magnitude of **F** and the perpendicular distance *or moment arm d*.

- If the force F is applied at an angle $\theta \neq 90^{\circ}$, as shown in Fig. b, then it will be more difficult to turn the bolt since the *moment arm* $d' = d \sin\theta$ will be smaller than d .

- If **F** is applied along the wrench, Fig. c , its moment arm will be zero since the line of action of **F** will intersect point O (the z axis). As a result, the moment of **F** about O is also zero and no turning can occur.

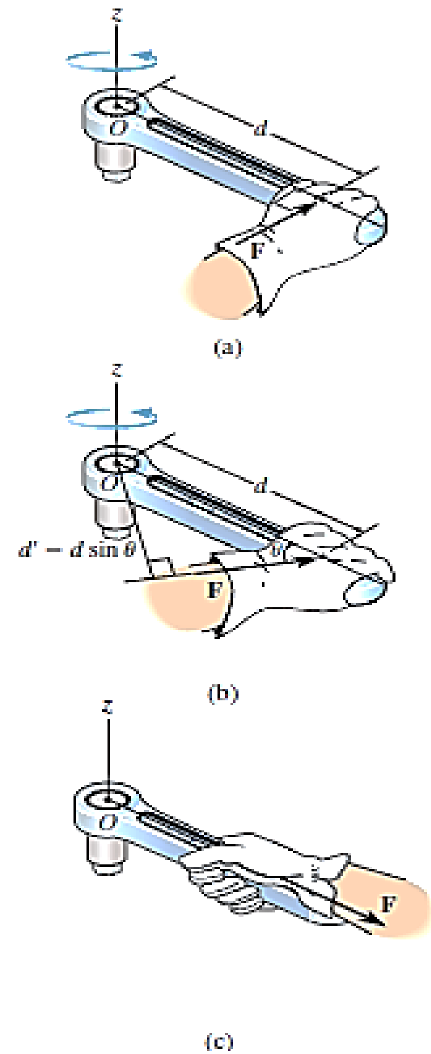


Fig. 14

- The moment M_O about point O , or about an axis passing through O and perpendicular to the plane, is a *vector quantity* since it has a specified magnitude and direction.

- The magnitude of **M_O** is

$$\boldsymbol{M_o = Fd}$$

Moment arm or *perpendicular distance*

Moment magnitude

N . m or lb . ft.

- Moment arm is defined as the perpendicular distance between axis of rotation and the line of action of force.

❖ Resultant Moment.

- For two-dimensional problems, where all the forces lie within the *x*–*y* plane.
- Fig. 15 , the resultant moment $(\mathbf{M}_R)_O$ about point ***O*** can be determined by finding the algebraic sum of the moments caused by all the forces in the system.

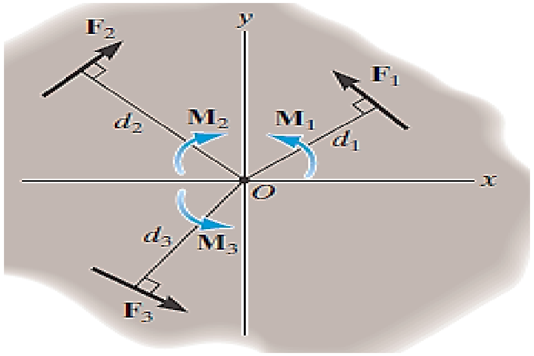


Fig. 15

- Positive moments as counterclockwise since they are directed along the positive.
- Clockwise moments will be negative.

- Doing this, the directional sense of each moment can be represented by a plus or minus sign.
- Using this sign convention, the resultant moment in Fig.15 is therefore

$$\circlearrowleft + (M_R)_o = \sum Fd; \quad = (M_R)_O = F_1 d_1 - F_2 d_2 + F_3 d_3$$

- If the numerical result of this sum is a positive scalar, $(M_R)_O$ will be a counterclockwise moment .
- if the result is negative, $(M_R)_O$ will be a clockwise moment.

Ex.1. For each case illustrated in Fig.17, determine the moment of the force about point O.

Sol.

- The line of action of each force is extended as a dashed line in order to establish the moment arm d.
- Also illustrated is the tendency of rotation of the member as caused by the force. Furthermore, the orbit of the force about O is shown as a colored curl. Thus,

❖ For fig.17a : $M_O = (100\text{ N})(2\text{ m}) = 200\text{ N.m}$

❖ For fig.17b : $M_O = (40\text{ lb})(4\text{ ft} + 2\cos 30^o\text{ ft}) = 229\text{ lb.ft}$

❖ For fig.17c : $M_O = (60\text{ lb})(1\sin 45^o\text{ ft}) = 42.4\text{ lb.ft}$

❖ For fig.17d : $M_O = (7\text{ kN})(4\text{ m} - 1\text{ m}) = 21.0\text{ kN.m}$

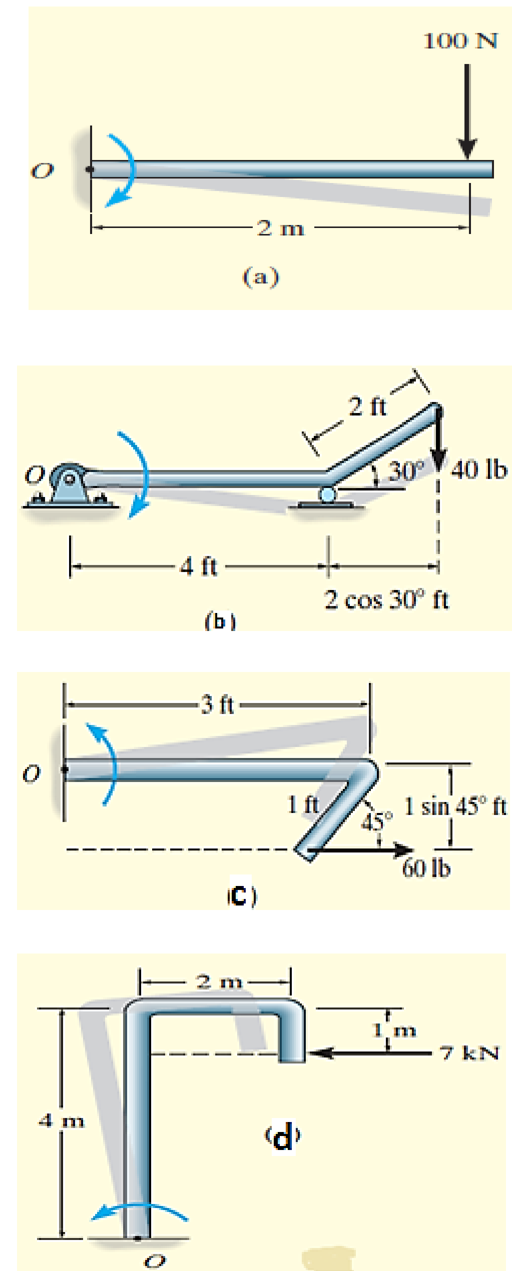


Fig.17

Ex.2. Determine the resultant moment of the four forces acting on the rod shown in Fig.18 about point O .

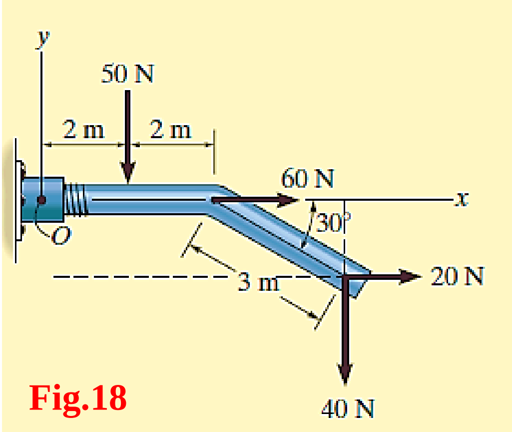


Fig.18

Sol.

Assuming that positive moments act in the +**k** direction, i.e., counterclockwise, we have

$$\circlearrowleft +(M_R)_o = \sum Fd;$$

$$\circlearrowleft (M_R)_O = -50\text{ N}(2\text{ m}) + 60\text{ N}(0) + 20\text{ N}(3\sin 30^{\circ}\text{ m}) - 40\text{ N}(4\text{ m} + 3\cos 30^{\circ}\text{ m})$$

$$(M_R)_O = -334\text{ N.m} = 334\text{ N.m} \circlearrowright$$

❖ Principle of moments:

- principle of moments , which is sometimes referred to as Varignon's theorem.

- It states that the moment of a force about a point is equal to the sum of the moments of the components of the force about the point .

- For example, consider the moments of the force **F** and two of its components about point O. Fig. 19 .

- Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

- The moments

$$M_O = r \times F = r \times (F_1 + F_2) = r \times F_1 + r \times F_2$$

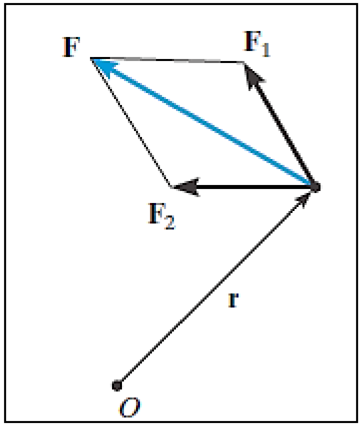


Fig. 19

- For two-dimensional problems, Fig. 20 , we can use the principle of moments by resolving the force into its rectangular components and then determine the moment using a scalar analysis

$$M_O = F_x y - F_y x$$

- This method is generally easier than finding the same moment using

$$M_O = Fd\ .$$

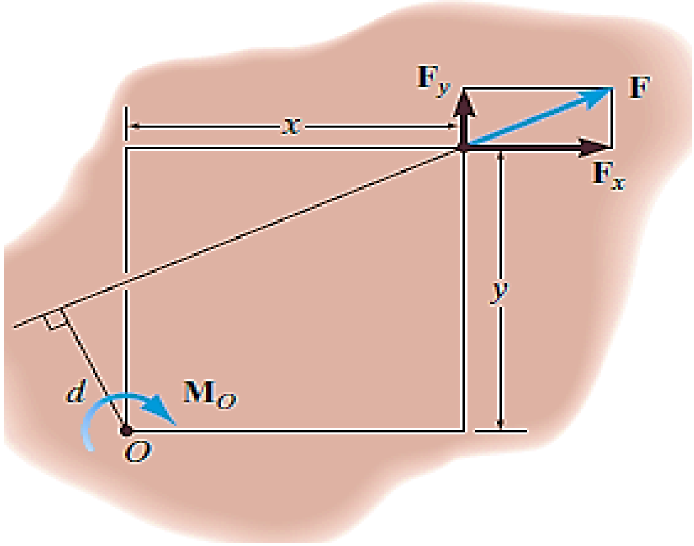


Fig. 20

Ex.3.Determine the moment of the force in Fig.21a about point O.

Sol. I

The moment arm d in Fig.21a can be found from trigonometry.

d = (3 m) sin 75° = 2.898 m

M_O = Fd = (5 kN)(2.898 m) = 14.5 kN. m

Sol. II

The x and y components of the force are indicated in Fig. 21b . Considering counterclockwise moments as positive, and applying the principle of moments, we have

$$M_o = F_x y - F_y x$$

$$= -(5\cos 45^\circ\ kN)(3\sin 30^\circ\ m) - (5\sin 45^\circ\ kN)(3\cos 30^\circ\ m)$$

$$= -14.5\ kN.m = 14.5\ kN.m$$

Sol. III

- The x and y axes can be set parallel and perpendicular to the rod's axis as shown in Fig. 21 c .
- Here $\mathbf{F}_x$ produces no moment about point O since its line of action passes through this point. Therefore,

$$+M_o = -Fy\ dx \quad = -(5\sin 75\ kN)(3\ m)$$

$$= -14.5\ kN.m = 14.5\ kN.m$$

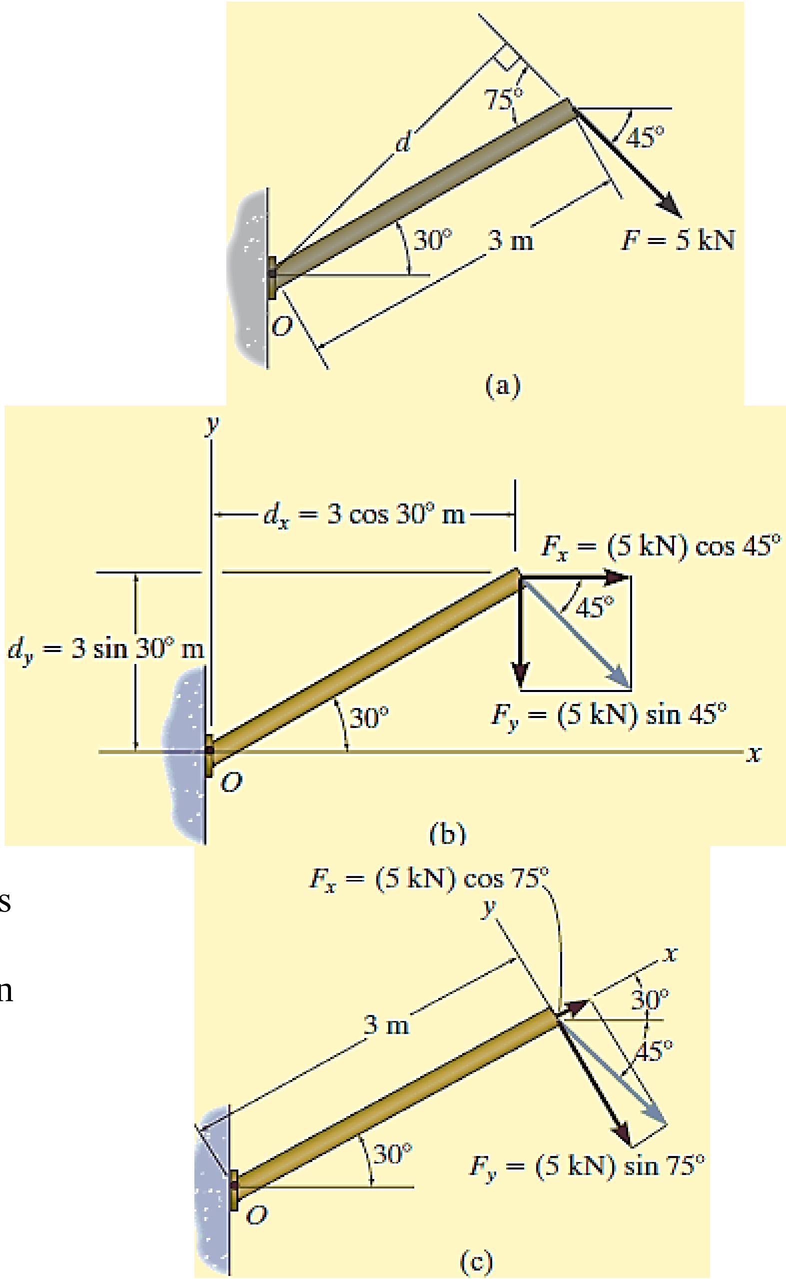


Fig.21

Ex.4. Force **F** acts at the end of the angle bracket in Fig. 22a. Determine the moment of the force about point O.

Sol.

The force is resolved into its x and y components, Fig.22b, then

$$M_o = F_x y - F_y x$$

$$+ M_o = 400 \sin 30^\circ \, N(0.2\, m) - 400 \cos 30^\circ \, N(0.4\, m)$$

$$= -98.6\, N.m = 98.6\, N.m$$

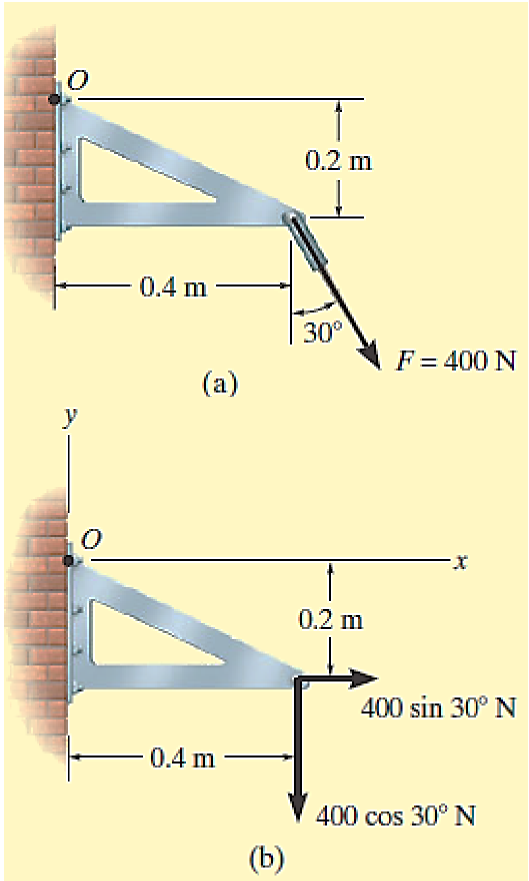


Fig. 22

❖ Moment of a Couple:

- A couple is defined as two parallel forces that have the same magnitude, but opposite directions, and are separated by a perpendicular distance d , Fig.23.

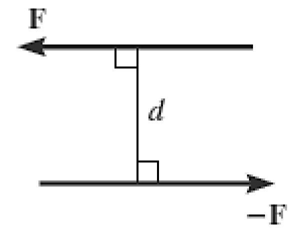


Fig.23.

- Since the resultant force is zero, the only effect of a couple is to produce an actual rotation, or if no movement is possible, there is a tendency of rotation in a specified direction.

- The moment produced by a couple is called a *couple moment*. We can determine its value by finding the sum of the moments of both couple forces about any arbitrary point.

- in Fig. 24 , position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are directed from point O to points A and B lying on the line of action of **-F** and **F** . The couple moment determined about O is therefore

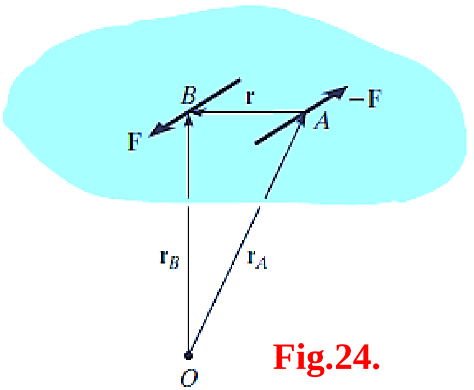


Fig.24.

$$M = r_B \times F + r_A \times -F = (r_B - r_A) \times F$$

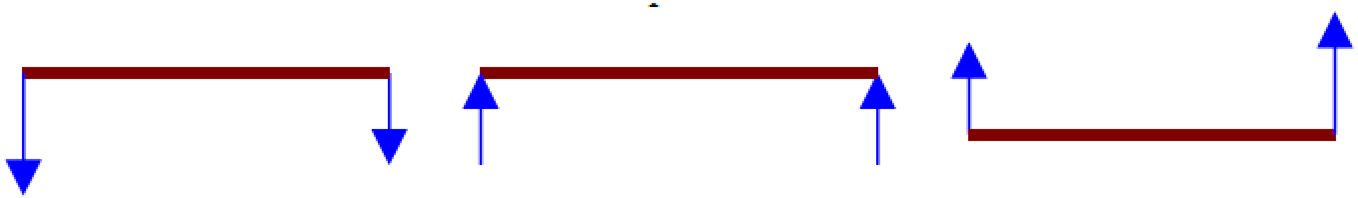
- However $\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}$ or $\boldsymbol{r} = \boldsymbol{r}_B - \boldsymbol{r}_A$

- so that

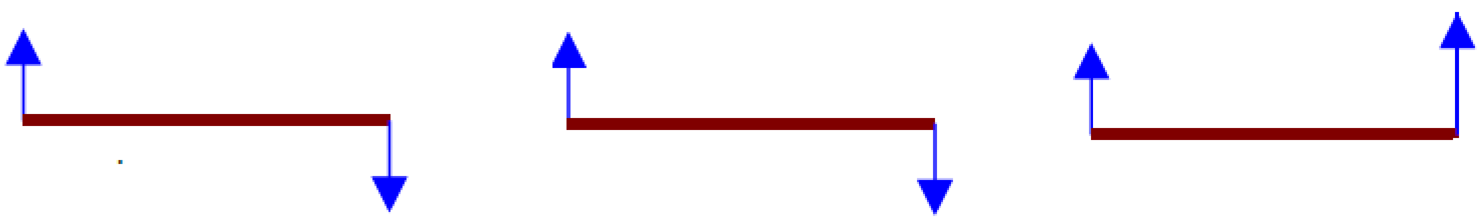
$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

- There are two types of parallel forces as discussed as under

- Like parallel forces
  When two parallel forces acing in such away that their directions remain same are called like parallel forces



- Un like parallel forces
  When two parallel forces acing in such away that their directions are opposite to each other called un like parallel forces



Scalar Formulation.

- The moment of a couple, **M** , Fig.25 , is defined as having a *magnitude* of

$$M = Fd$$

where

- $\boldsymbol{F}$ is the magnitude of one of the forces .
- $\boldsymbol{d}$ is the perpendicular distance or moment arm between the forces.

- In all cases, $\boldsymbol{M}$ will act perpendicular to the plane containing these forces.

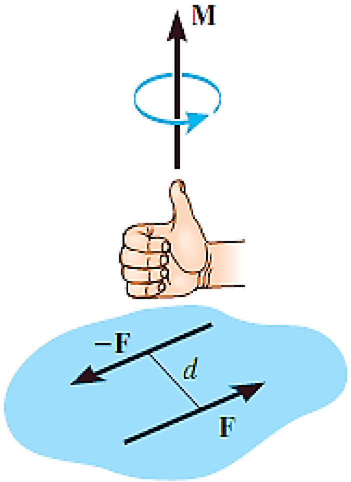


Fig.25

Vector Formulation.

- The moment of a couple can also be expressed by the vector cross product

$$M = r \times F$$

- Application of this equation is easily remembered if one thinks of taking the moments of both forces about a point lying on the line of action of one of the forces.

- For example, if moments are taken about point A in Fig. 24 , the moment of **-F** is zero about this point, and the moment of **F** is defined from above equation . Therefore, in the formulation **r** is crossed with the force **F** to which it is directed.

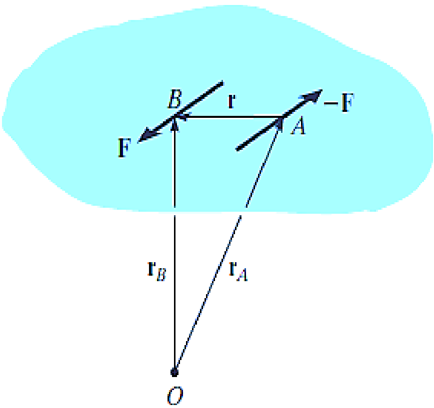


Fig.24

❑ Equivalent Couples.

- If two couples produce a moment with the same magnitude and direction , then these two couples are equivalent .

- As shown in Fig. 26 are equivalent because each couple moment has a magnitude of **M = 30 N(0.4 m) = 40 N(0.3 m) = 12 N .m.**

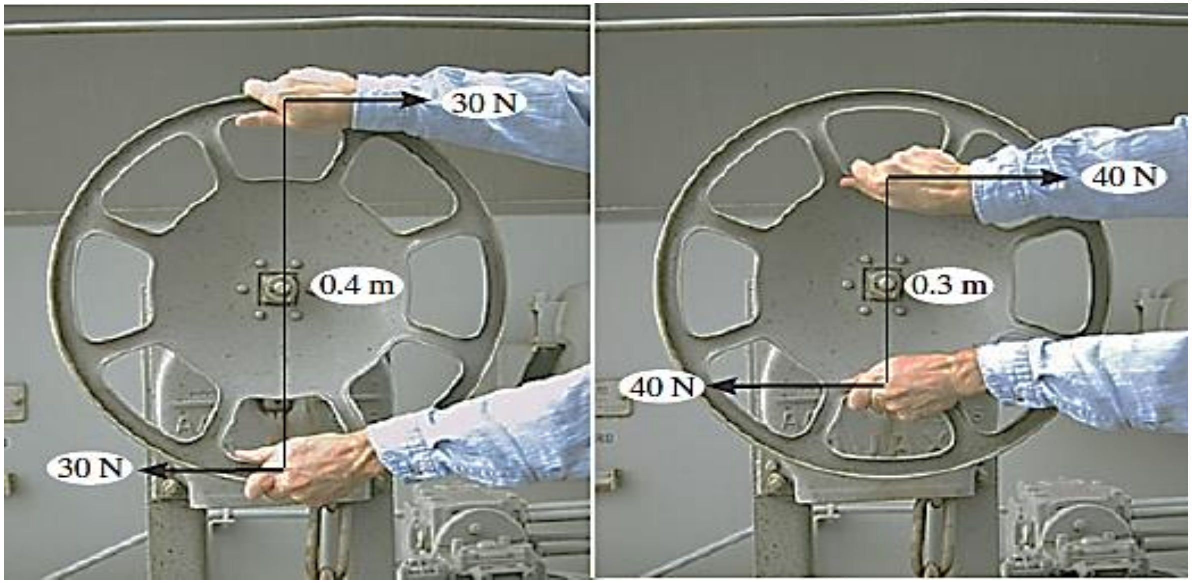


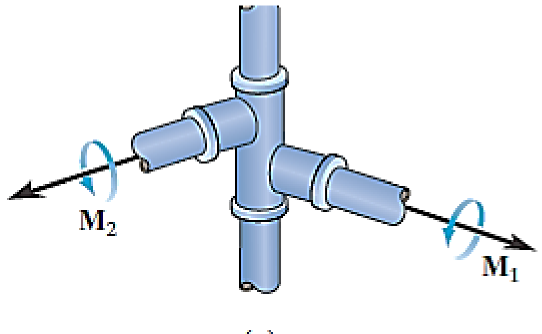
Fig. 26

- Notice that larger forces are required in the second case to create the same turning effect because the hands are placed closer together.

Resultant Couple Moment.

- Since couple moments are vectors, their resultant can be determined by vector addition.

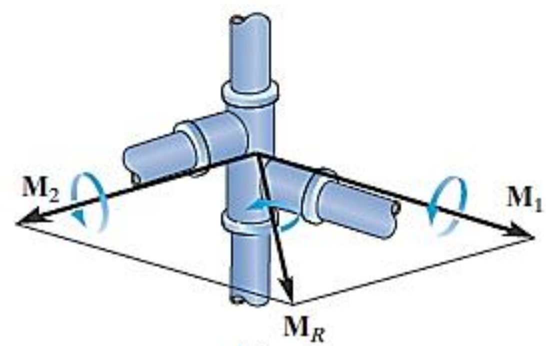
$$M_R = M_1 + M_2$$



(a)

- If more than two couple moments act on the body, we may generalize this concept and write the vector resultant as

$$M_R = \sum (r \times F)$$



(b)

Ex.5.Determine the resultant couple moment of the three couples acting on the plate in Fig. 27 .

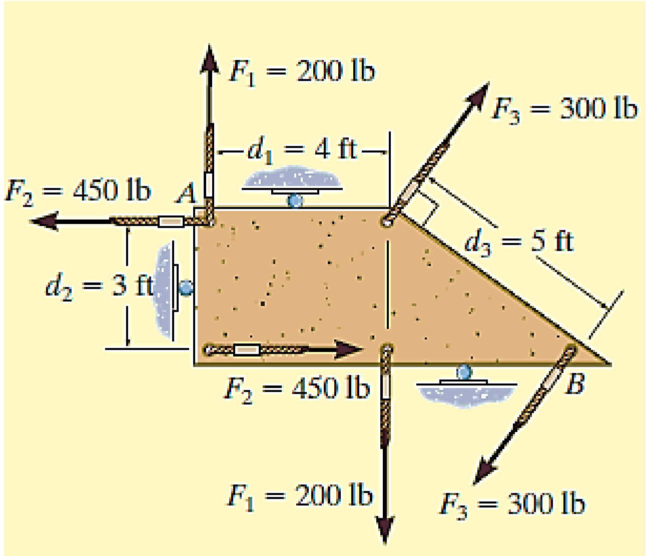


Fig. 27

Sol.

- As shown the perpendicular distances between each pair of couple forces are $d_1 = 4$ ft, $d_2 = 3$ ft, and $d_3 = 5$ ft .

- Considering counterclockwise couple moments as positive, we have

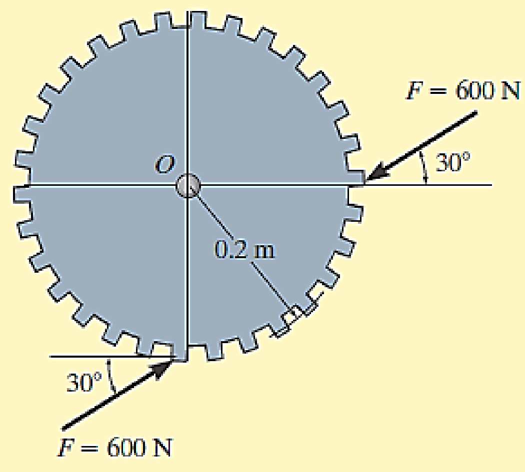
$$+M_R = M; \quad M_R = -F_1d_1 + F_2d_2 - F_3d_3$$

$$= -(200\ lb)(4\ ft) + (450\ lb)(3\ ft) - (300\ lb)(5\ ft) = -950\ lb.ft = 950\ lb.ft$$

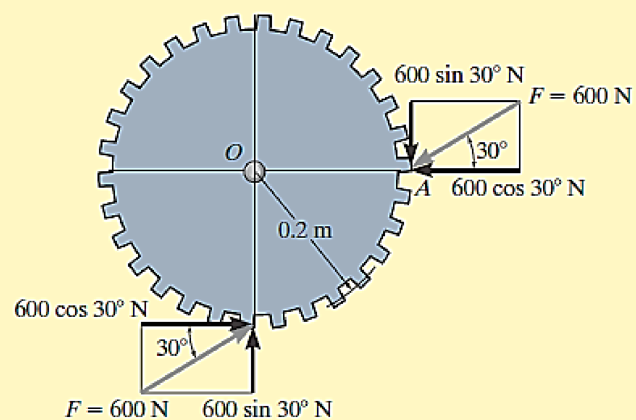
- The negative sign indicates that $\mathbf{M}_R$ has a clockwise rotational sense.

Ex.6.Determine the magnitude and direction of the couple moment acting on the gear in Fig.28a.

Sol.



(a)



(b)

- The easiest solution requires resolving each force into its components as shown in Fig.28b.
- The couple moment can be determined by summing the moments of these force components about any point, for example, the center O of the gear or point A.
- If we consider counterclockwise moments as positive, we have

$$\circlearrowleft + M = \sum M_o;$$

$$M = (600\cos 30^\circ N)(0.2\,m) - (600\sin 30^\circ N)(0.2\,m)$$
$$= 43.9\,N.m \circlearrowleft$$

or
$$\circlearrowleft + M = \sum M_A;$$

$$M = (600\cos 30^\circ N)(0.2\,m) - (600\sin 30^\circ N)(0.2\,m)$$
$$= 43.9\,N.m \circlearrowleft$$

This positive result indicates that M has a counterclockwise rotational sense

➢ The Free-Body Diagram.

- To apply the equation of equilibrium, we must account for all the known and unknown forces ($\sum F = 0$) which act on the particle.

- The best way to do this is to think of the particle as isolated and "free" from its surroundings. A drawing that shows the particle with all the forces that act on it is called a free-body diagram (FBD).

The 5-kg plate is suspended by two straps *A* and *B*. To find the force in each strap we should consider the free-body diagram of the plate. As noted, the three forces acting on it form a concurrent force system.

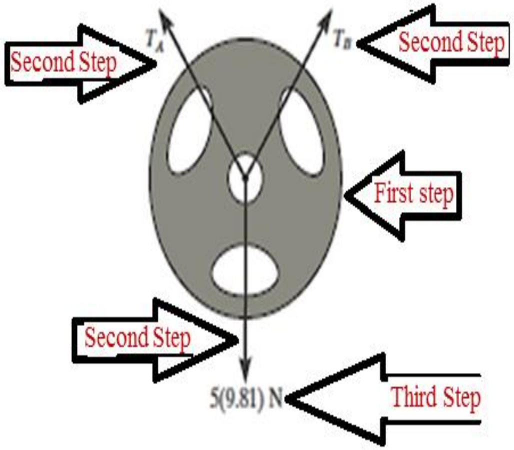
❑ Procedure for Drawing a Free-Body Diagram

To construct a free-body diagram, the following three steps are necessary.

- First step: Draw Outlined Shape.
  Imagine the particle to be isolated or cut "free" from its surroundings by drawing its outlined shape.

- Second Step: Show All Forces.
  Indicate on this sketch all the forces that act on the particle . These forces can be active forces , which tend to set the particle in motion, or they can be reactive forces which are the result of the constraints or supports that tend to prevent motion. To account for all these forces, it may be helpful to trace around the particle's boundary, carefully noting each force acting on it.

- Third Step: Identify Each Force.
  The forces that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are unknown.



➢ Coplanar Force Systems.

- If a particle is subjected to a system of coplanar forces that lie in the *x–y plane*, as in Fig. 1, then each force can be resolved into its ***i*** *and* ***j*** *components*.

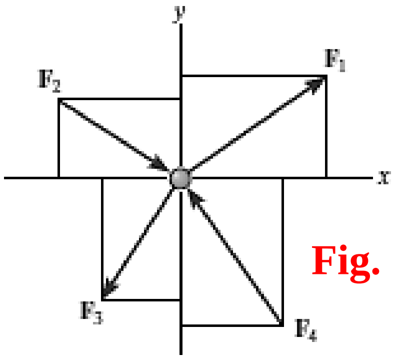


Fig. 1

- For equilibrium, these forces must sum to produce a zero force resultant,

$$\sum \mathrm{F} = 0$$

$$\sum F_x i + \sum F_y j = 0$$

- For this vector equation to be satisfied, the resultant force’s x and y components must both be equal to zero.

$$\sum F_x = 0 \qquad , \qquad \sum F_y = 0$$

- At applying each of the two equations of equilibrium, we must account for the sense of direction of any component by using an algebraic sign which corresponds to the arrowhead direction of the component along the x or y-axis.

- It is important to note that if a force has an unknown magnitude , then the arrowhead sense of the force on the free-body diagram can be assumed. Then if the solution yields a negative scalar , this indicates that the sense of the force is opposite to that which was assumed.

Ex.1. Determine the tension in cables BA and BC necessary to support the 60-kg cylinder in Fig.2 a .

Sol.

Free-Body Diagram.

- Due to equilibrium, the weight of the cylinder causes the tension in cable BD to be TBD = 60(9.81) N , Fig. 2 b .
- The forces in cables BA and BC can be determined by investigating the equilibrium of ring B . Its free-body diagram is shown in Fig. 2 c . The magnitudes of TA and TC are unknown, but their directions are known.

Equations of Equilibrium.

- Applying the equations of equilibrium along the x and y axes, we have

$$\overset{+}{\rightarrow} \sum F_x = 0; \qquad T_C \cos 45^\circ - \left(\frac{4}{5}\right) T_A = 0 \ldots\ldots\ldots(1)$$

$$+\uparrow \sum F_y = 0; \qquad T_C \sin 45^\circ - \left(\frac{3}{5}\right) T_A - 60(9.81)N = 0 \ldots\ldots\ldots(2)$$

- Equation (1) can be written as $\boldsymbol{T_A = 0.8839 T_C}$. Substituting this into Eq. (2) yields

$$T_C \sin 45^\circ - \left(\frac{3}{5}\right) \mathbf{0.8839 T_C} - 60(9.81)N = 0$$

- so that $T_C = 475.66\ N = 476\ N$

- Substituting this result into either Eq. (1) or Eq. (2), we get

$$T_A = 420\ N$$

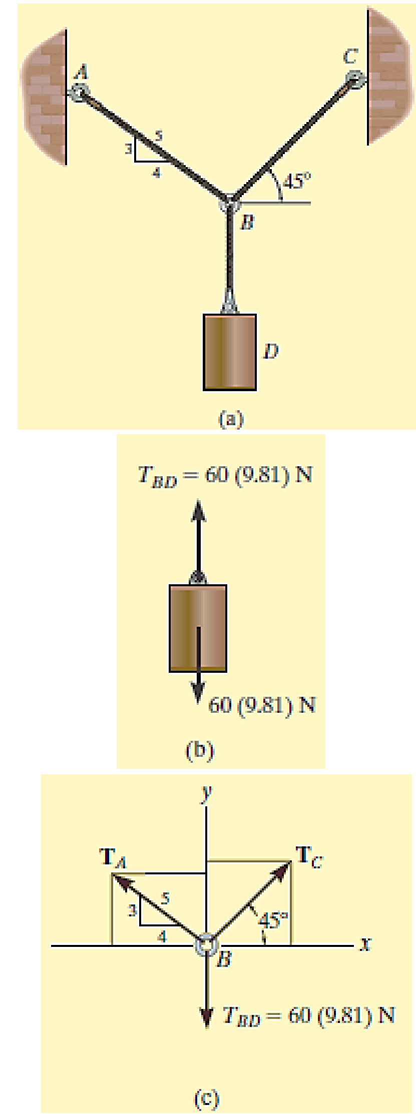


Fig.2

Ex.2. The 200-kg crate in Fig. 3a is suspended using the ropes AB and AC . Each rope can withstand a maximum force of 10 kN before it breaks. If AB always remains horizontal, determine the smallest angle θ to which the crate can be suspended before one of the ropes breaks.

Sol.

Free-Body Diagram.

- We will study the equilibrium of ring A . There are three forces acting on it, Fig. 3 b . The magnitude of $\mathbf{F}_D$ is equal to the weight of the crate, i.e., F_D = 200 (9.81) N = 1962 N < 10 kN .

Equations of Equilibrium.

- Applying the equations of equilibrium along the x and y axes,

$$\xrightarrow{+} \sum F_x = 0; \qquad -F_C \cos\theta + F_B = 0; \quad F_C = \frac{F_B}{\cos\theta} \qquad (1)$$

$$+\uparrow \sum F_y = 0; \qquad F_C \sin\theta - 1962 \text{ N} = 0 \qquad (2)$$

- From Eq. (1), F_C is always greater than F_B since cos $\theta \leq 1$. Therefore, rope AC will reach the maximum tensile force of 10 kN *before rope* AB . Substituting F_C = 10 kN into Eq. (2), we get

$$[\,10(10^3)\text{N}]\sin\theta - 1962N = 0$$
$$\theta = sin^{-1}(0.1962) = 11.31°$$

- The force developed in rope AB can be obtained by substituting the values for θ and F_C into Eq. (1).

$$10(10^3)\text{N} = \frac{F_B}{\cos 11.31°} \rightarrow F_B = 9.81 \text{ kN}$$

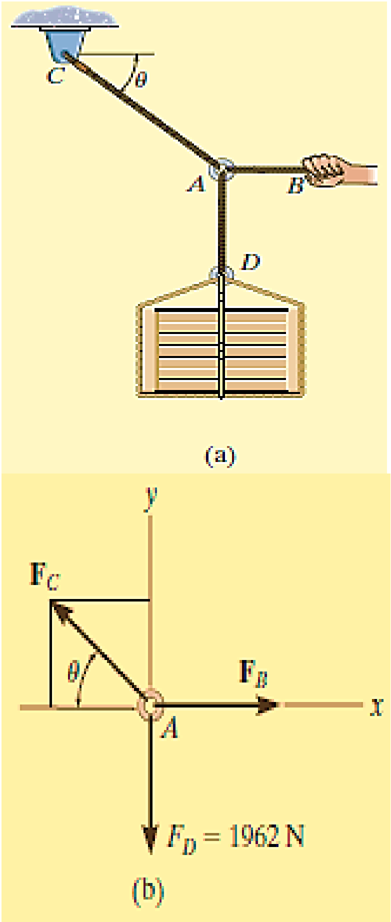


Fig. 3.

❖ Equilibrium of a Rigid Body

- In this section, we will develop both the necessary and sufficient conditions for the equilibrium of the rigid body in Fig. 4 a . As shown, this body is subjected to an external force and couple moment system that is the result of the effects of gravitational, electrical, magnetic, or contact forces caused by adjacent bodies.

- The internal forces caused by interactions between particles within the body are not shown in this figure because these forces occur in equal but opposite collinear pairs and hence will cancel out, a consequence of Newton’s third law.

- The force and couple moment system acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point O on or off the body.

- Fig. 4 b . If this resultant force and couple moment are both equal to zero, then the body is said to be in equilibrium . Mathematically, the equilibrium of a body is expressed as

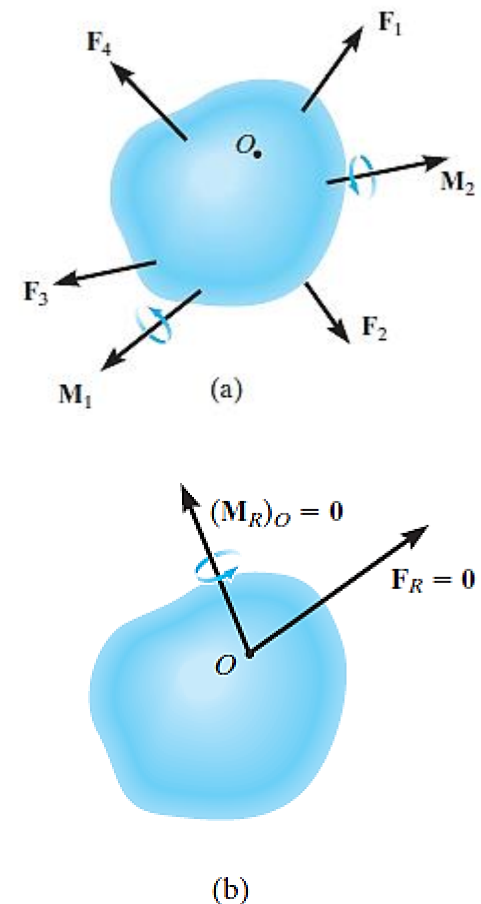


Fig. 4

$$F_R = \sum F = 0 \qquad (M_R)_o = \sum M_o = 0$$

- The first of these equations states that the sum of the forces acting on the body is equal to zero . The second equation states that the sum of the moments of all the forces in the system about point O , added to all the couple moments, is equal to zero .

➢ Free-Body Diagrams.

- Successful application of the equations of equilibrium requires a complete specification of all the known and unknown external forces that act on the body.

- The best way to account for these forces is to draw a free-body diagram.

- This diagram is a sketch of the outlined shape of the body, which represents it as being isolated or "free" from its surroundings, i.e., a "free body."

- On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied.

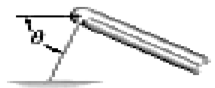
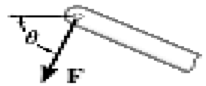
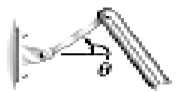
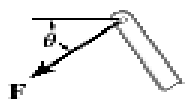
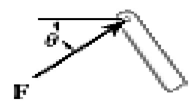
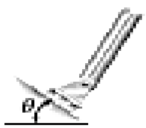
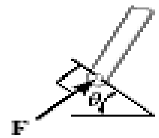
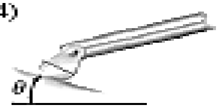
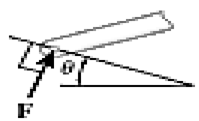
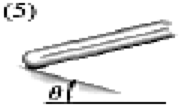
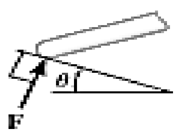
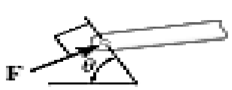
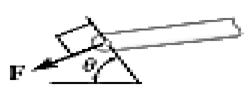
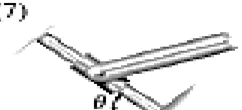
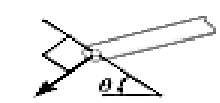
- We need to know the following things before knowing how to draw the F.B.D.

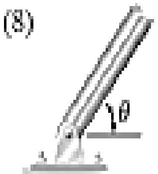
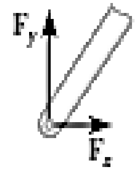
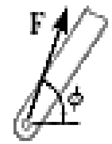
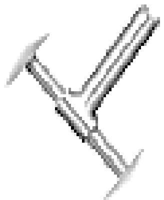
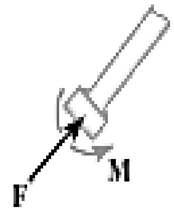
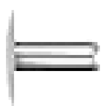
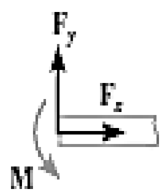
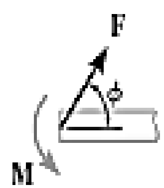
▪ Support Reactions.

- There are various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems.

- Table 1 lists a common types of supports for bodies subjected to coplanar force systems. (In all cases the angle θ is assumed to be known.) .

TABLE 1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge	 or 	Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support	 or 	Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

▪ External and Internal forces.

- Since a rigid body is a composition of particles, both external and internal loadings may act on it.

- Only the external loading are represented on the free body diagram because the net effect of the internal forces on the body is zero.

▪ Weight of a Body.

- When a body is subjected to a gravitational field, then it has a specified weight.

- The weight of the body is represented by a resultant force located at the center of gravity of the body.

$$W = m \times g \, , where \;\; g = 9.81 \frac{m}{s^2}$$

❖ Procedure for Analysis

To construct a free-body diagram for a rigid body or any group of bodies considered as a single system, the following steps should be performed:

- Draw Outlined Shape.

  Imagine the body to be *isolated* or cut "free" from its constraints and connections and draw (sketch) its outlined shape.

- Show All Forces and Couple Moments.

Identify all the known and unknown *external forces* and couple moments that *act on the body* . Those generally encountered are due to:

- Applied loadings,
- Reactions occurring at the supports or at points of contact with other bodies (see Table 1),
- The weight of the body.

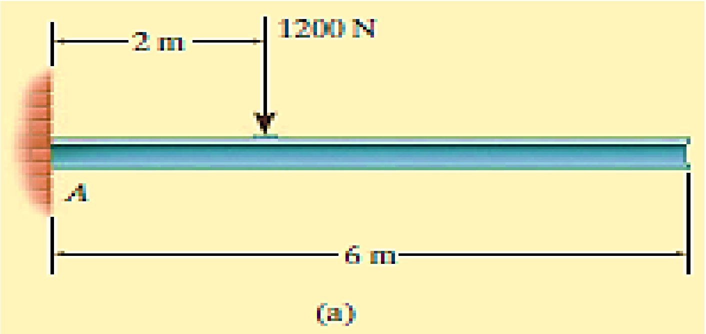
To account for all these effects, it may help to trace over the boundary, carefully noting each force or couple moment acting on it.

- Identify Each Loading and Give Dimensions.

- The forces and couple moments that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and direction angles of forces and couple moments that are unknown.
- Establish an x, y coordinate system so that these unknowns, A_x, A_y, etc., can be identified.
- Finally, indicate the dimensions of the body necessary for calculating the moments of forces.

Ex.3. Draw the free-body diagram of the uniform beam shown in Fig. 5 a . The beam has a mass of 100 kg.

Fig. 5a



Sol.

- The free-body diagram of the beam is shown in Fig. 5 b .
- Since the support at A is fixed, the wall exerts three reactions on the beam , denoted as A_x, A_y, and M_A .
- The magnitudes of these reactions are unknown , and their sense has been assumed .
- The weight of the beam, W = 100(9.81) N = 981 N, acts through the beam's center of gravity G , which is 3 m from A since the beam is uniform.

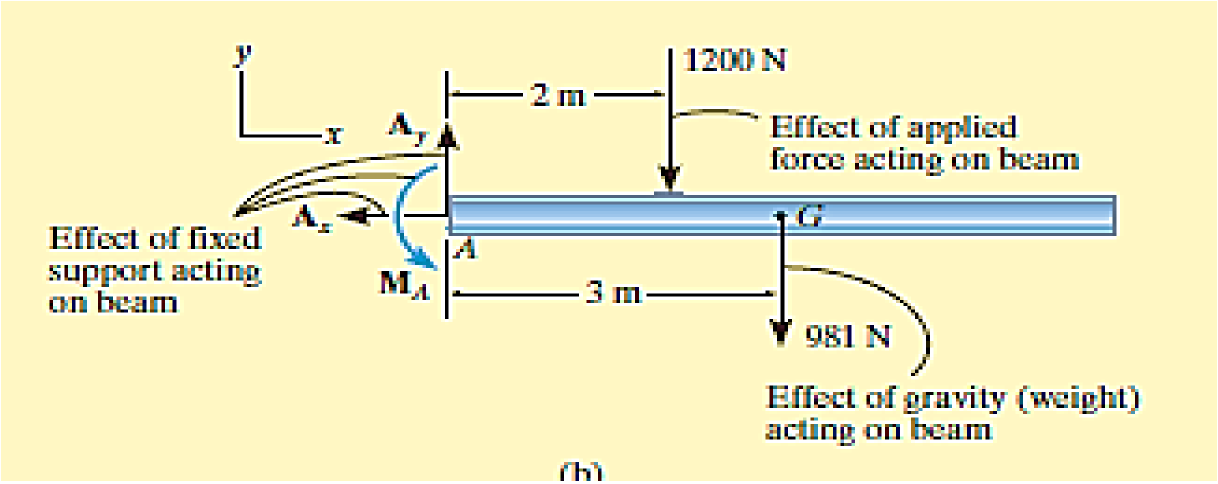


Fig. 5b

➢ Equations of Equilibrium

- Two equations which are both necessary and sufficient for the equilibrium of a rigid body, namely, $\sum F = 0$ and $\sum M_o = 0$.

- When the body is subjected to a system of forces, which all lie in the $x - y$ plane, then the forces can be resolved into their x and y components.

- Consequently, the conditions for equilibrium in two dimensions are

$$\sum F_x = 0 \quad , \quad \sum F_y = 0 \quad , \sum M_o = 0$$

- Here $\sum F_x$ and $\sum F_y$ represent, respectively, the algebraic sums of the x and y components of all the forces acting on the body, and $\sum M_o$ represents the algebraic sum of the couple moments and the moments of all the force components about the z axis, which is perpendicular to the x–y plane and passes through the arbitrary point O .

❖ **Procedure for Analysis**

Coplanar force equilibrium problems for a rigid body can be solved using the following procedure.

- Free-Body Diagram.

  - Establish the x, y coordinate axes in any suitable orientation.
  - Draw an outlined shape of the body.
  - Show all the forces and couple moments acting on the body.
  - Label all the loadings and specify their directions relative to the x or y axis. The sense of a force or couple moment having an unknown magnitude but known line of action can be assumed .
  - Indicate the dimensions of the body necessary for computing the moments of forces.

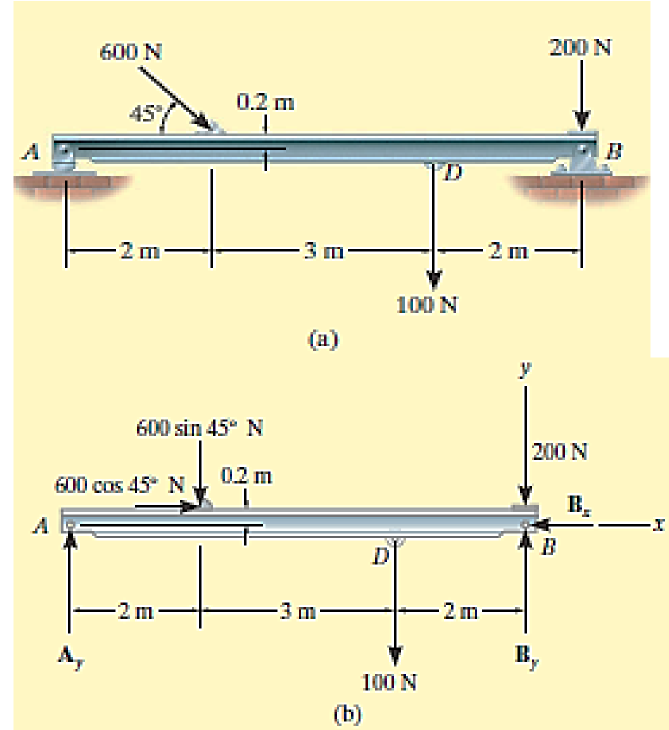
- Equations of Equilibrium.

  - Apply the moment equation of equilibrium, $\sum M_o = 0$, about a point (O) that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about O, and a *direct solution* for the third unknown can be determined.

  - When applying the force equilibrium equations, $\sum F_x = 0$ and $\sum F_y = 0$, orient the x and y axes along lines that will provide the simplest resolution of the forces into their x and y components.

  - If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, this indicates that the sense is opposite to that which was assumed on the free-body diagram.

Ex.5. Determine the horizontal and vertical components of reaction on the beam caused by the pin at B and the rocker at A as shown in Fig.7a . Neglect the weight of the beam.



Sol.

❖ Free-Body Diagram.

Identify each of the forces shown on the free-body diagram of the beam, Fig. 7 *b* . (See Example 3 .) For simplicity, the 600-N force is represented by its *x* and *y* components as shown in Fig. 7 *b* .

❖ Equations of Equilibrium.

- Summing forces in the x direction yields

$$\overset{+}{\rightarrow}\sum F_x = 0; \quad 600\cos 45^\circ N - B_x = 0 \longrightarrow B_x = 424N$$

- A direct solution for Ay can be obtained by applying the moment equation $\sum M_B = 0$ about point B

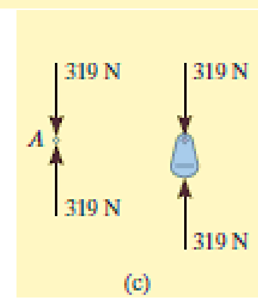


Fig.7

$$\circlearrowleft + \sum M_B = 0,$$

$$100\ N\ (2\ m) + (600 \sin 45^\circ\ N)(5\ m) - (600 \cos 45^\circ N)(0.2\ m) - A_y\ (7\ m) = 0$$

$$A_y = 319\ N$$

- Summing forces in the y direction, using this result, gives

$$+\uparrow \sum F_y = 0; \quad 319\ N - 600 \sin 45^\circ\ N - 100\ N - 200\ N + B_y = 0 \longrightarrow B_y = 405\ N$$

NOTE: Remember, the support forces in Fig. 7 b are the result of pins that act on the beam. The opposite forces act on the pins. For example, Fig. 7 c shows the equilibrium of the pin at A and the rocker.

Ex.6.The cord shown in Fig.8a supports a force of 100 lb and wraps over the frictionless pulley. Determine the tension in the cord at C and the horizontal and vertical components of reaction at pin A .

Sol.

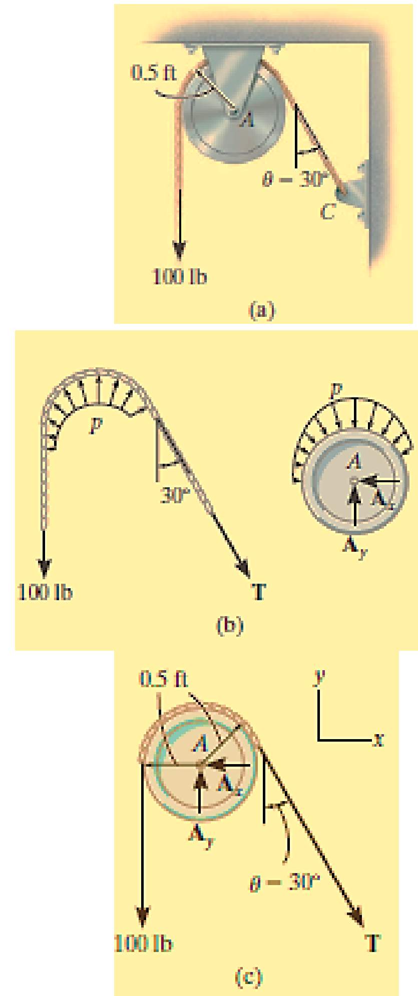


Fig.8

❖ Free-Body Diagram.

- The free-body diagrams of the cord and pulley are shown in Fig.8b .
- Note that the principle of action, equal but opposite reaction must be carefully observed when drawing each of these diagrams: the cord exerts an unknown load distribution p on the pulley at the contact surface, whereas the pulley exerts an equal but opposite effect on the cord.
- For the solution, however, it is simpler to combine the free-body diagrams of the pulley and this portion of the cord, so that the distributed load becomes internal to this “system” and is therefore eliminated from the analysis, Fig.8 c .

❖ Equations of Equilibrium.

- Summing moments about point A to eliminate A_x and A_y, Fig. 8c , we have

$$\circlearrowleft +\sum M_A = 0, \quad 100\ lb\,(0.5\ ft) - T(0.5\ ft) = 0 \longrightarrow T = 100\ lb$$

- Using this result,

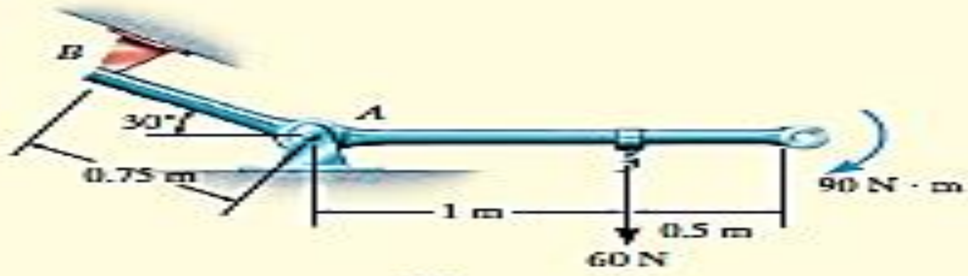
$$\overset{+}{\rightarrow}\sum F_x = 0; \quad -A_x + 100\ \sin 30^\circ\ lb = 0 \longrightarrow A_x = 50.0\ lb$$

$$+\uparrow \sum F_y = 0; \quad A_y - 100\ lb - 100\cos 30^\circ\ lb = 0 \longrightarrow A_y = 187\ lb$$

NOTE: From the moment equation, it is seen that the tension remains *constant* as the cord passes over the pulley. (This of course is true for *any angle* θ at which the cord is directed and for *any radius* r of the pulley.)

Ex.7.

The member shown in Fig. 5–14*a* is pin connected at *A* and rests against a smooth support at *B*. Determine the horizontal and vertical components of reaction at the pin *A*.



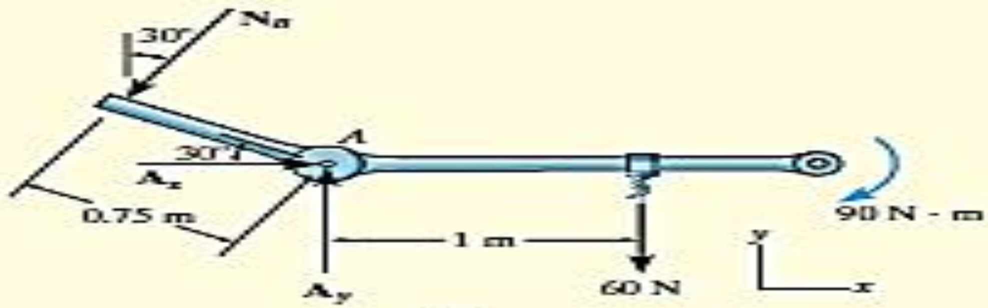


Fig. 5–14

SOLUTION

Free-Body Diagram. As shown in Fig. 5–14*b*, the reaction $\mathbf{N}_B$ is perpendicular to the member at *B*. Also, horizontal and vertical components of reaction are represented at *A*.

Equations of Equilibrium. Summing moments about *A*, we obtain a direct solution for N_B.

$$\circlearrowleft + \Sigma M_A = 0; \quad -90\ \text{N}\cdot\text{m} - 60\ \text{N}(1\ \text{m}) + N_B(0.75\ \text{m}) = 0$$

$$N_B = 200\ \text{N}$$

Using this result,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad A_x - 200 \sin 30^\circ\ \text{N} = 0$$

$$A_x = 100\ \text{N} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \quad A_y - 200 \cos 30^\circ\ \text{N} - 60\ \text{N} = 0$$

$$A_y = 233\ \text{N} \qquad \textit{Ans.}$$

Ex.8. The box wrench in Fig. 5–15*a* is used to tighten the bolt at A. If the wrench does not turn when the load is applied to the handle, determine the torque or moment applied to the bolt and the force of the wrench on the bolt.

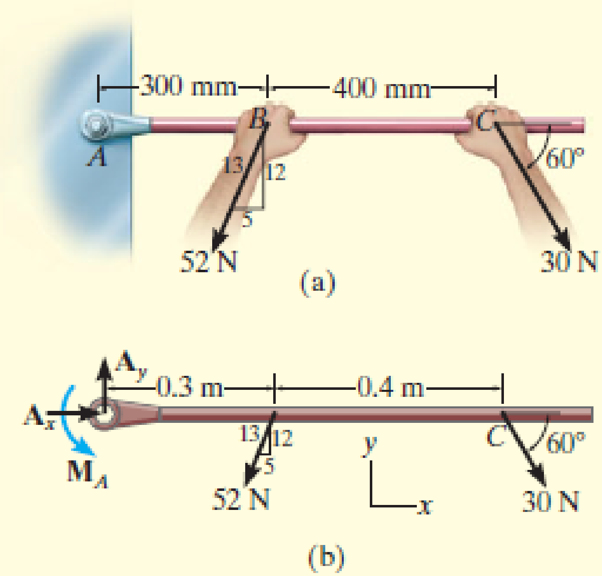


Fig. 5–15

SOLUTION

Free-Body Diagram. The free-body diagram for the wrench is shown in Fig. 5–15*b*. Since the bolt acts as a "fixed support," it exerts force components $\mathbf{A}_x$ and $\mathbf{A}_y$ and a moment $\mathbf{M}_A$ on the wrench at A.

Equations of Equilibrium.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad A_x - 52\left(\tfrac{5}{13}\right)\text{ N} + 30\cos 60^\circ\text{ N} = 0$$

$$A_x = 5.00\text{ N} \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad A_y - 52\left(\tfrac{12}{13}\right)\text{ N} - 30\sin 60^\circ\text{ N} = 0$$

$$A_y = 74.0\text{ N} \qquad \textit{Ans.}$$

$$\circlearrowleft +\Sigma M_A = 0; \quad M_A - \left[52\left(\tfrac{12}{13}\right)\text{N}\right](0.3\text{ m}) - (30\sin 60^\circ\text{ N})(0.7\text{ m}) = 0$$

$$M_A = 32.6\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

Friction

❖ **Friction** is a force that resists the movement of two contacting surfaces that slide relative to one another.

- This force always acts tangent to the surface at the points of contact and is directed so as to oppose the possible or existing motion between the surfaces.
- In this lecture , we will study the effects of dry friction , which is sometimes called Coulomb friction.
- Dry friction occurs between the contacting surfaces of bodies when there is no lubricating fluid.

The heat generated by the abrasive action of friction can be noticed when using this grinder to sharpen a metal blade.

➢ Theory of Dry Friction..

- The theory of dry friction can be explained by considering the effects caused by pulling horizontally on a block of uniform **weight W** which is resting on a rough horizontal surface that is non rigid or deformable, Fig.1 a .

- The upper portion of the block can be considered rigid.

- As shown on the free-body diagram of the block, Fig. 1 b , the floor exerts an uneven distribution of both **normal force** ΔN_n and **frictional force** ΔF_n along the contacting surface.

- For equilibrium, the normal forces must act upward to balance the block's **weight W** , and the frictional forces act to the left to prevent the **applied force P** from moving the block to the right.

- Close examination of the contacting surfaces between the floor and block reveals how these frictional and normal forces develop, Fig.1 c.

- It can be seen that many microscopic irregularities exist between the two surfaces and, as a result, **reactive forces** ΔR_n are developed at each point of contact.

- As shown, each reactive force contributes both a **frictional component** ΔF_n **and a normal component** ΔN_n.

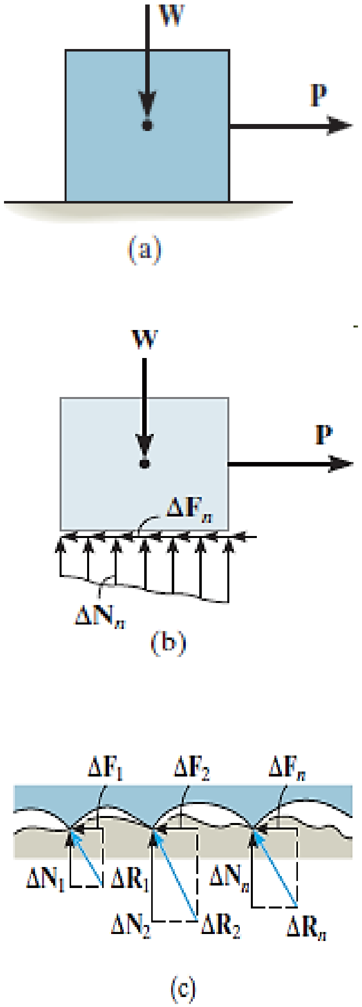


Fig. 1

➢ Equilibrium.

- The effect of the distributed normal and frictional loadings is indicated by their resultants *N* and *F* on the free-body diagram, Fig. 2.
- Notice that *N* acts a distance *x* to the right of the line of action of **W** , Fig.2.
- This location, which coincides with the centroid or geometric center of the normal force distribution in Fig.1 b , is necessary in order to balance the "tipping effect" caused by P .

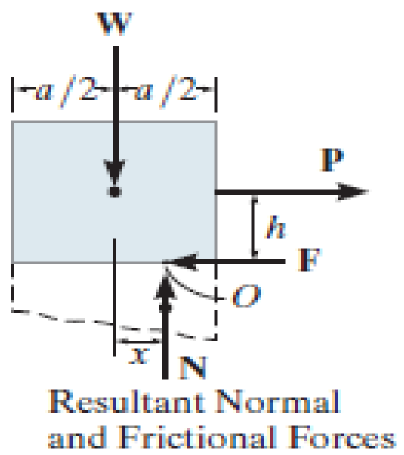


Fig. 2

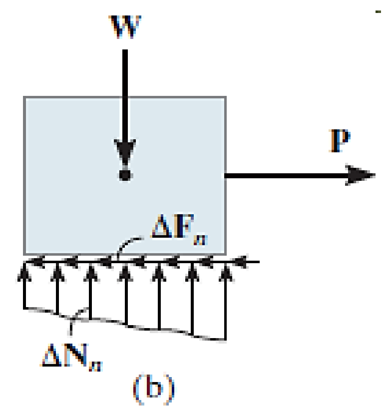


Fig.1 b

- For example, if P is applied at a height h from the surface, Fig. 2, then moment equilibrium about point O is satisfied if $Wx = \mathrm{Ph}$ or $x = \frac{Ph}{w}$.

➢ Impending Motion.

- In cases where the surfaces of contact are rather "slippery," the frictional force F may not be great enough to balance P , and consequently the block will tend to slip.

- As **P** is slowly increased, **F** correspondingly increases until it attains a certain maximum value Fs, called the limiting static frictional force , Fig. 3.

- When this value is reached, the block is in unstable equilibrium since any further increase in P will cause the block to move.

- limiting static frictional force Fs is directly proportional to the resultant normal force N . Expressed mathematically,

$$F_s = \mu_s N$$

- where the constant of proportionality, μ_s, is called the coefficient of static friction .

- Angle of static friction $\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1}\mu_s$

- Typical values for μ_s, are given in Table 1 .

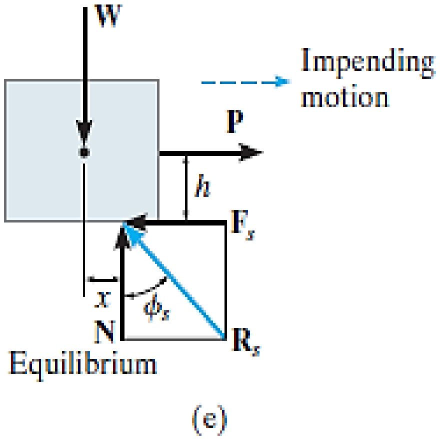


Fig. 3.

Table 1 Typical Values for μ_s

Contact Materials	Coefficient of Static Friction (μ_s)
Metal on ice	0.03–0.05
Wood on wood	0.30–0.70
Leather on wood	0.20–0.50
Leather on metal	0.30–0.60
Aluminum on aluminum	1.10–1.70

➢ Motion.

- If the magnitude of P acting on the block is increased so that it becomes slightly greater than F_s, the frictional force at the contacting surface will drop to a smaller value F_k, called the kinetic frictional force .

- The block will begin to slide with increasing speed, Fig.4a . As this occurs, the block will "ride" on top of these peaks at the points of contact, as shown in Fig.4b . The continued breakdown of the surface is the dominant mechanism creating kinetic friction.

- The magnitude of the kinetic friction force is directly proportional to the magnitude of the resultant normal force, expressed mathematically as

$$F_k = \mu_k N$$

- μ_k, is called the coefficient of kinetic friction . Typical values for μ_k are approximately 25 percent smaller than those listed in Table1for μ_s.

- Angle of kinetic friction $\phi_k = \tan^{-1}\left(\frac{F_k}{N}\right) = \tan^{-1}\left(\frac{\mu_k N}{N}\right) = \tan^{-1}\mu_k$

- By comparison, $\phi_s \geq \phi_k$.

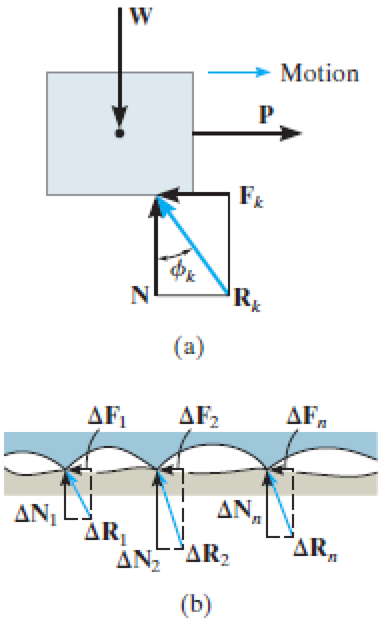


Fig.4

➢ Procedure for Analysis

Equilibrium problems involving dry friction can be solved using the following procedure.

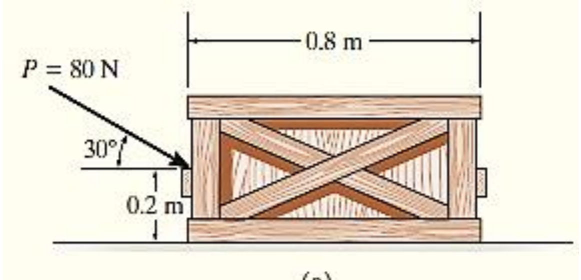
❖ Free-Body Diagrams.

- Draw the necessary free-body diagrams, and unless it is stated in the problem that impending motion or slipping occurs, *always* show the frictional forces as unknowns (i.e., *do not assume* $F = \mu N$).
- Determine the number of unknowns and compare this with the number of available equilibrium equations.
- If there are more unknowns than equations of equilibrium, it will be necessary to apply the frictional equation at some, if not all, points of contact to obtain the extra equations needed for a complete solution.
- If the equation $F = \mu N$ is to be used, it will be necessary to show **F** acting in the correct sense of direction on the free-body diagram.

❖ Equations of Equilibrium and Friction.

- Apply the equations of equilibrium and the necessary frictional equations (or conditional equations if tipping is possible) and solve for the unknowns.

Ex.1. The uniform crate shown in Fig.5a has a mass of 20 kg. If a force P = 80 N is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction is $\mu_s = 0.3$.

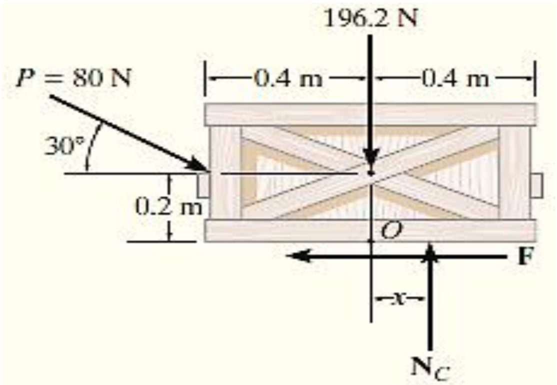


(a)

Sol.

❖ Free-Body Diagram.

- As shown in Fig.5b , the resultant normal force **NC** must act a distance **x** from the crate's center line in order to counteract the tipping effect caused by P.
- There are three unknowns, **F** , **NC**, and **x** , which can be determined strictly from the three equations of equilibrium.



(b)

Fig.5

❖ Equations of Equilibrium.

$$\overset{+}{\rightarrow}\sum F_x = 0; \quad 80\cos 30^\circ\text{N} - \text{F} = 0 \rightarrow F = 69.3\text{ N}$$

$$+\uparrow \sum F_y = 0; -80\sin 30^\circ\text{N} + \text{NC} - 196.2\text{ N} = 0 \rightarrow \text{NC} = 236.2\text{ N}$$

$$\circlearrowleft + \sum M_o = 0; \; 80\sin 30^\circ N(0.4\,m) - 80\cos 30^\circ N(0.2\,m) + NC(x) = 0 \rightarrow x = -0.00908\text{ m} = -9.08\text{ mm}$$

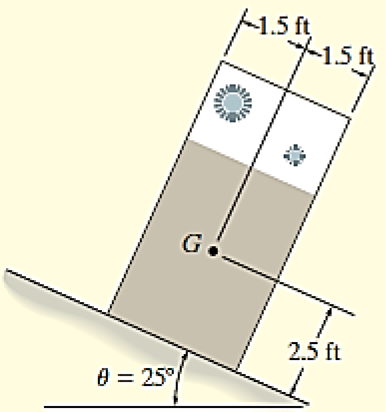
- Since x is negative it indicates the *resultant* normal force acts (slightly) to the *left* of the crate's center line. No tipping will occur since $x < 0.4$ m.
- Also, the *maximum* frictional force which can be developed at the surface of contact is $F_{max} = \mu_s NC = 0.3(236.2\,N) = 70.9\,N$. Since $F = 69.3$ N < 70.9 N, the crate will *not slip* , although it is very close to doing so.

Ex.2. It is observed that when the bed of the dump truck is raised to an angle of $\theta = 25°$ the vending machines will begin to slide off the bed, Fig. 6 a . Determine the static coefficient of friction between a vending machine and the surface of the truckbed.

(a)

Sol.

An idealized model of a vending machine resting on the truckbed is shown in Fig. 6b . The dimensions have been measured and the center of gravity has been located. We will assume that the vending machine weighs W .



(b)

❖ Free-Body Diagram.

As shown in Fig. 6c , the dimension x is used to locate the position of the resultant normal force N . There are four unknowns, N, F ,μ_s, and x .

❖ Equations of Equilibrium.

$$+\searrow\Sigma F_x = 0; \qquad W \sin 25° - F = 0 \qquad (1)$$

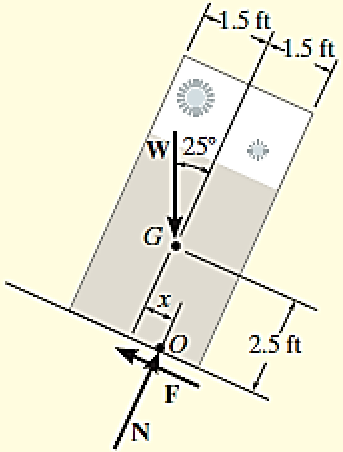
$$+\nearrow\Sigma F_y = 0; \qquad N - W \cos 25° = 0 \qquad (2)$$

$$\circlearrowleft +\Sigma M_O = 0; \quad -W \sin 25°(2.5 \text{ ft}) + W \cos 25°(x) = 0 \qquad (3)$$

Since slipping impends at $\theta = 25°$, using Eqs. 1 and 2, we have

$$F_s = \mu_s N; \qquad W \sin 25° = \mu_s(W \cos 25°)$$

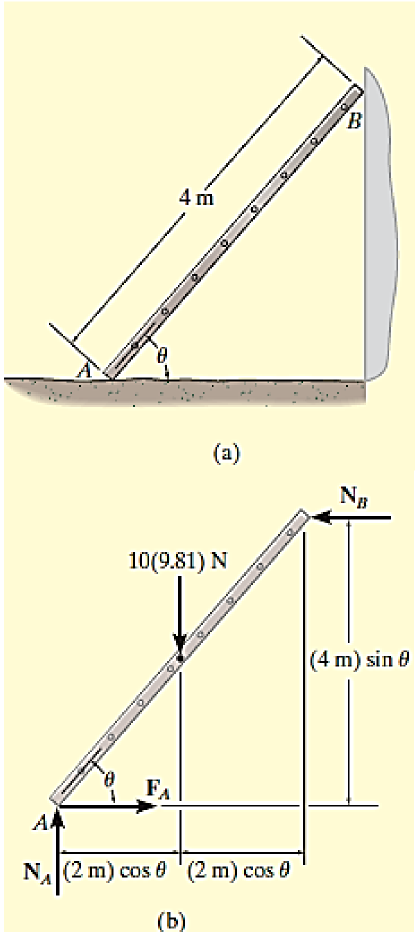
$$\mu_s = \tan 25° = 0.466$$



(c)

from Eq. 3 , we find $x = 1.17$ ft. Since 1.17 ft < 1.5 ft, indeed the vending machine will slip before it can tip as observed in Fig. 6 a .

Ex.3. The uniform 10-kg ladder in Fig.7 a rests against the smooth wall at B , and the end A rests on the rough horizontal plane for which the coefficient of static friction is $\mu_s = 0.3$. Determine the angle of inclination θ of the ladder and the normal reaction at B if the ladder is on the verge of slipping.



Sol.

❖ Free-Body Diagram.

As shown on the free-body diagram, Fig.7b , the frictional force F_A must act to the right since impending motion at A is to the left.

❖ Equations of Equilibrium and Friction.

Since the ladder is on the verge of slipping, then $F_A = \mu_S N_A = 0.3 N_A$. By inspection, N_A can be obtained directly.

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 10(9.81)\text{ N} = 0 \qquad N_A = 98.1\text{ N}$$

Using this result, $F_A = 0.3(98.1\text{ N}) = 29.43\text{ N}$. Now N_B can be found.

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 29.43\text{ N} - N_B = 0$$

$$N_B = 29.43\text{ N} = 29.4\text{ N}$$

Finally, the angle θ can be determined by summing moments about point A.

$$\circlearrowleft + \Sigma M_A = 0; \qquad (29.43\text{ N})(4\text{ m})\sin\theta - [10(9.81)\text{ N}](2\text{ m})\cos\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = 1.6667$$

$$\theta = 59.04^\circ = 59.0^\circ$$

Center of Gravity and Centroid

❖ Center of Gravity, Center of Mass, and the Centroid of a Body.

❑ Center of Gravity.

- A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW , Fig. 1 a.
- These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the center of gravity, G , Fig. 1 b.
- The weight of the body is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \sum F_z \quad ; \qquad W = \int dW$$

- The location of the center of gravity, measured from the y $axis$, is determined by equating the moment of W about the y $axis$, Fig.1b , to the sum of the moments of the weights of the particles about this same axis. If dW is located at point ($\tilde{x}$, $\tilde{y}$, $\tilde{z}$), Fig.1 a , then

$$(M_R)_y = \sum M_y \qquad \bar{x}\,\mathrm{W} = \int \tilde{x}\ \mathrm{dW}$$

- Similarly, if moments are summed about the x $axis$,

$$(M_R)_x = \sum M_x \qquad \bar{y}\,\mathrm{W} = \int \tilde{y}\ \mathrm{dW}$$

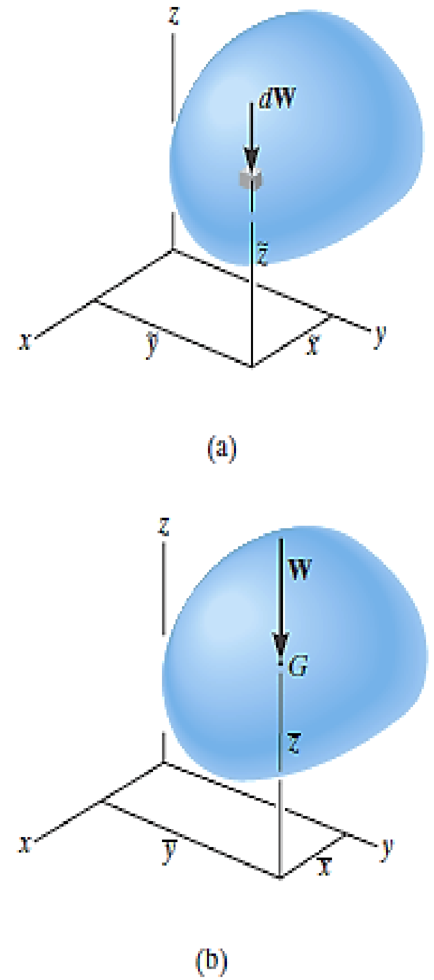


Fig. 1.

- Finally, imagine that the body is fixed within the coordinate system and this system is rotated 90° about the y $axis$, Fig. 2. Then the sum of the moments about the y $axis$ gives.

$$(M_R)_y = \sum M_y \qquad \bar{z}\, W = \int \tilde{z}\ \mathrm{dW}$$

- Therefore, the location of the center of gravity G with respect to the x , y , z $axes$ becomes

$$\boxed{\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \quad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \quad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}} \qquad \ldots\ldots (1)$$

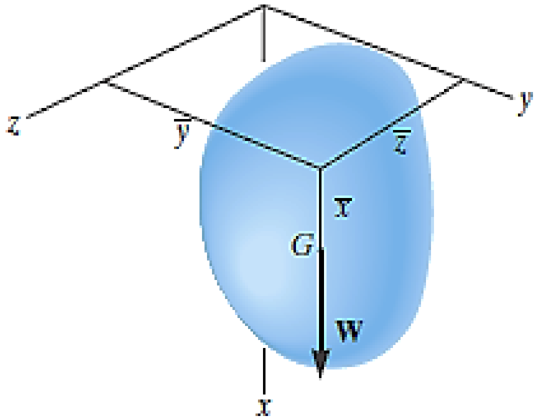


Fig. 2.

Here

$\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the center of gravity G , $Fig.1\ b$.

$\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are the coordinates of each particle in the body, $Fig.1\ a$.

❑ Center of Mass of a Body.

- In order to study the dynamic response or accelerated motion of a body, it becomes important to locate the body's center of mass C_m, Fig.3 .
- This location can be determined by substituting $dW = g\,dm$ into Eqs. (1) . Since g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\,dm}{\int dm} \quad \ldots\ldots (2)$$

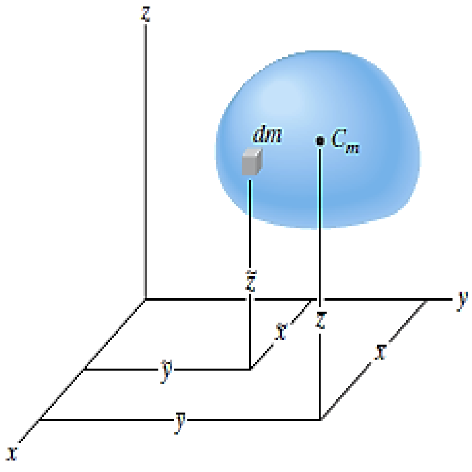


Fig. 3.

❑ Centroid of a Volume.

- If the body in Fig. 4 is made from a homogeneous material, then its density ρ will be constant.
- Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$.
- Substituting this into Eqs. 2 and canceling out ρ, we obtain formulas that locate the centroid C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} \quad \ldots\ldots (3)$$

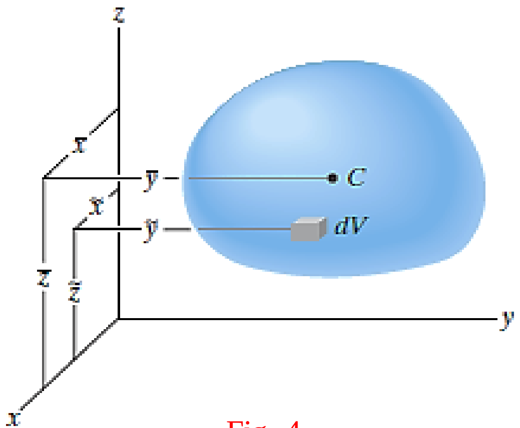


Fig. 4.

❑ Centroid of an Area.

- If an area lies in the $x-y$ *plane* and is bounded by the curve $y = f(x)$, as shown in $Fig. 5a$, then its centroid will be in this plane and can be determined from integrals similar to $Eqs. 3.$, namely,

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \qquad \ldots\ldots(4)$$

- These integrals can be evaluated by performing a single integration if we use a rectangular strip for the differential area element.

- For example, if a vertical strip is used, $Fig. 5b$, the area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = \frac{y}{2}$.

- If we consider a horizontal strip, $Fig. 5c$, then $dA = x\,dy$, and its centroid is located at $\tilde{x} = \frac{x}{2}$ and $\tilde{y} = y$.

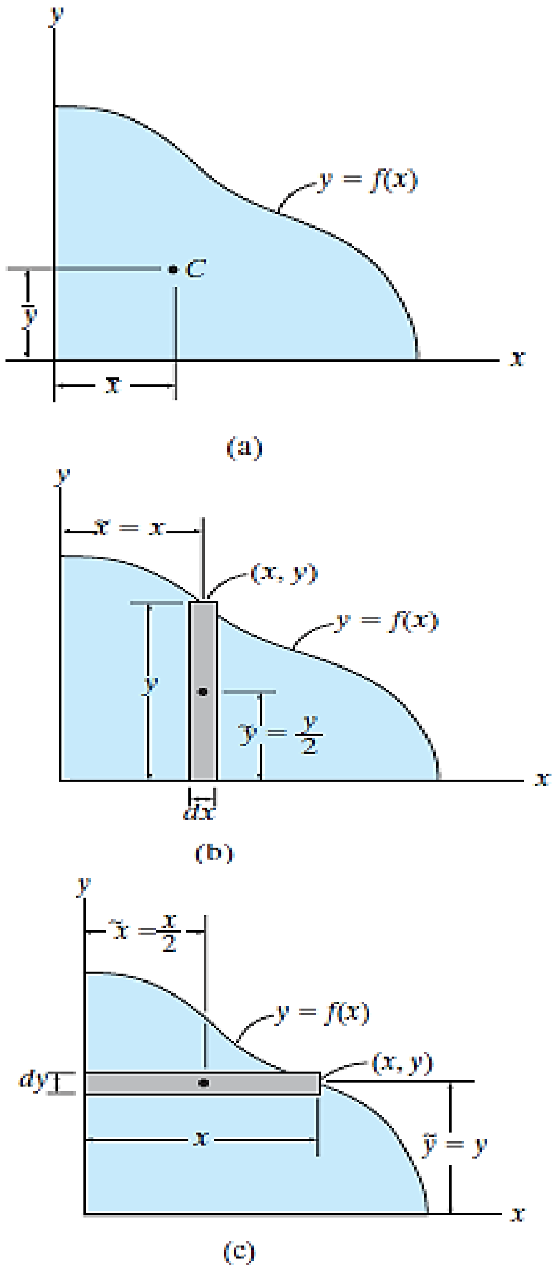


Fig. 5.

❑ Centroid of a Line.

- If a line segment (or rod) lies within the $x-y$ *plane* and it can be described by a thin curve $y = f(x)$, $Fig.6a$, then its centroid is determined from

$$\boxed{\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL}} \qquad \ldots\ldots(5)$$

- Here, the length of the differential element is given by the Pythagorean theorem,

$$dL = \sqrt{(dx)^2 + (dy)^2}$$

- which can also be written in the form

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

- or

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2}$$

$$= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

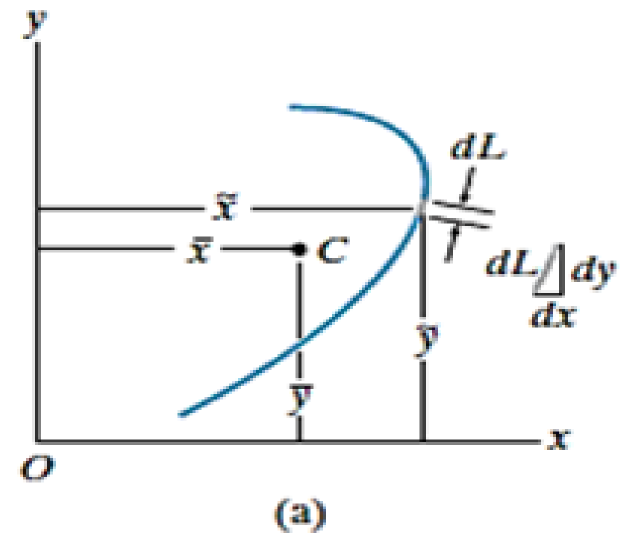


Fig.6a.

- For example, consider the rod in $Fig. 6b$, defined by $y = 2x^2$.

- The length of the element is $dL = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$

- Since $\frac{dy}{dx} = 4x$,

- Then $dL = \sqrt{1 + (4x)^2}\, dx$.

- The centroid for this element is located at

- $\tilde{x} = x$

- $\tilde{y} = y$

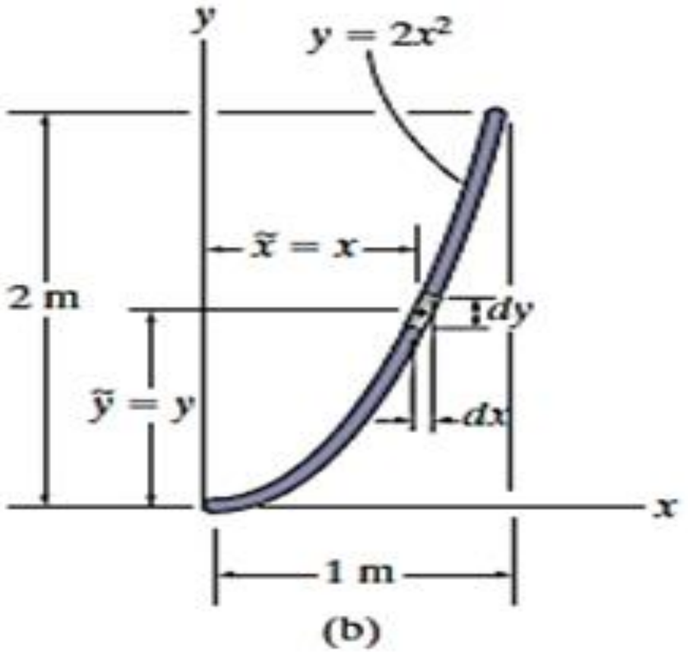


Fig.6b.

❖ Procedure for Analysis

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

❑ Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For lines the element is represented by a differential line segment of length dL .
- For areas the element is generally a rectangle of area dA , having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV , having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

❑ Size and Moment Arms.

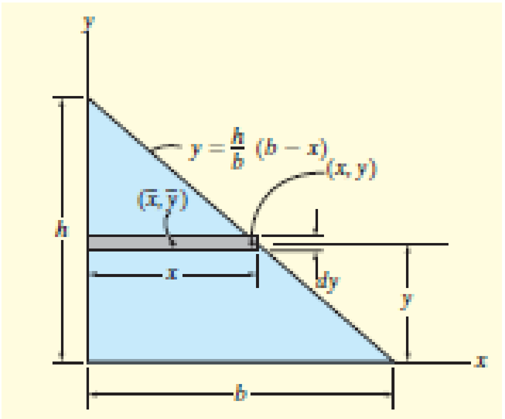
- Express the length dL , area dA , or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}, \tilde{y}, \tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

❑ Integrations.

- Substitute the formulations for $\tilde{x}, \tilde{y}, \tilde{z}$ and dL, dA , or dV into the appropriate equations ($Eqs. 1$ $through$ 5).
- Express the function in the integrand in terms of the same variable as the differential thickness of the element.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.

Ex.1. Determine the distance $\tilde{y}$ measured from the x *axis* to the centroid of the area of the triangle shown in $Fig.7$.

Sol.



$Fig.7$

- Differential Element.
  - Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), $Fig.7$.
- Area and Moment Arms.
  - The area of the element is $dA = x\,dy = \frac{b}{h}(h - y)dy$ and its centroid I located a distance $\tilde{y} = y$ from the x axis.
- Integrations.
  - Applying the second of Eqs. 4 and integrating with respect to y yields

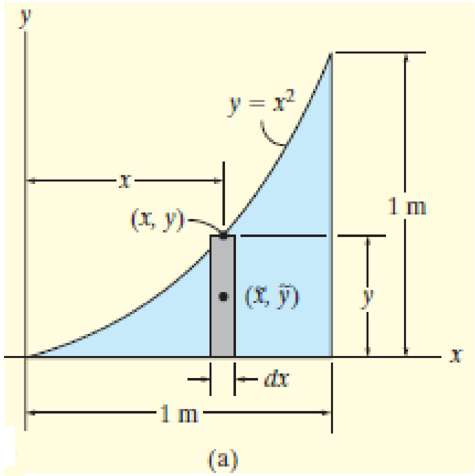
$$\bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{\int_0^h y\left[\frac{b}{h}(h - y)\, dy\right]}{\int_0^h \frac{b}{h}(h - y)\, dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} = \frac{h}{3}$$

  - NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

Ex.2. Locate the centroid of the area shown in $Fig.8a$.

Sol.I.

- Differential Element.
  - A differential element of thickness dx is shown in Fig.8a .
  - The element intersects the curve at the arbitrary point (x, y), and so it has a height y .



$Fig.8a$

- Area and Moment Arms.
  - The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x, \tilde{y} = \frac{y}{2}$.

- Integrations.
  - Applying $Eqs.4$ and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m}$$

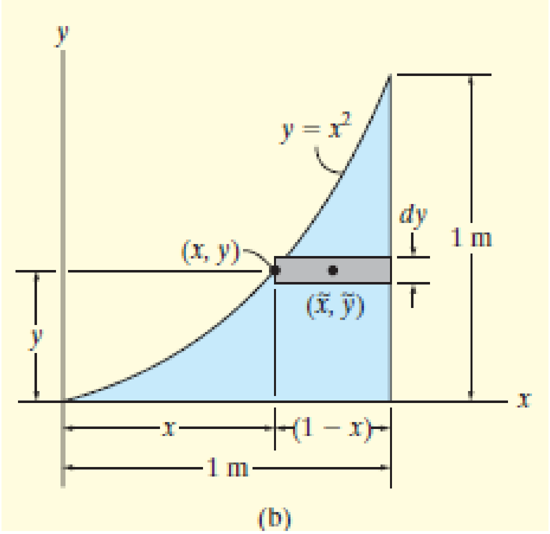
$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m}$$

Sol.II.

- Differential Element.
- The differential element of thickness dy is shown in $Fig.8b$. The element intersects the curve at the arbitrary point (x, y), and so it has a length $(1 - x)$.

- Area and Moment Arms.
- The area of the element is $dA = (1 - x)\, dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2}\right) = \frac{1 + x}{2}, \; \tilde{y} = y$$



$Fig.8b$.

- Integrations.
- Applying $Eqs.4$ and integrating with respect to y yields

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1 + x)/2](1 - x)\, dy}{\int_0^{1\text{ m}} (1 - x)\, dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1 - y)\, dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\, dy} = \frac{0.250}{0.333} = 0.75\text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1 - x)\, dy}{\int_0^{1\text{ m}} (1 - x)\, dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\, dy}{\int_0^{1\text{ m}} (1 - \sqrt{y})\, dy} = \frac{0.100}{0.333} = 0.3\text{ m}$$

- NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

Ex.4. Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 9–14.

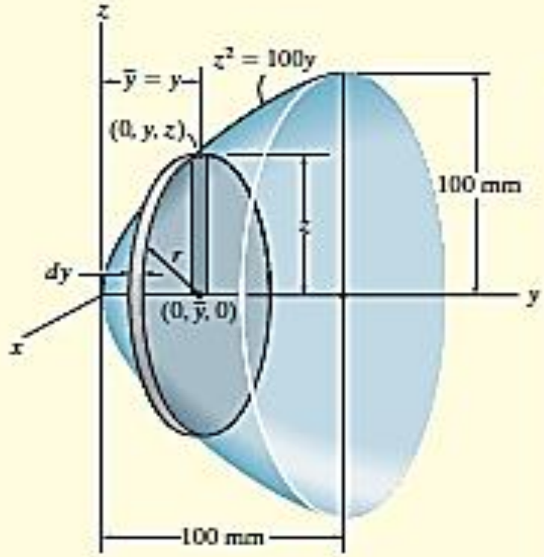


Fig. 9–14

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\,dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–3 and integrating with respect to y yields

$$\bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\,dy}{\int_0^{100\text{ mm}} (\pi z^2)\,dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\,dy}{100\pi \int_0^{100\text{ mm}} y\,dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

Ex.5. Determine the location of the center of mass of the cylinder shown in Fig. 9–15 if its density varies directly with the distance from its base, i.e., $\rho = 200z\ \text{kg/m}^3$.

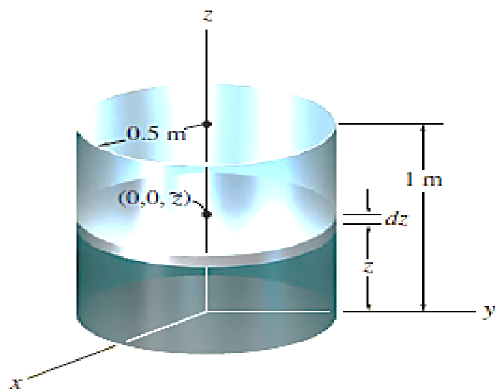


Fig. 9–15

SOLUTION

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Differential Element. A disk element of radius 0.5 m and thickness dz is chosen for integration, Fig. 9–15, since the *density of the entire element is constant* for a given value of z. The element is located along the z axis at the *arbitrary point* $(0, 0, z)$.

Volume and Moment Arm. The volume of the element is $dV = \pi(0.5)^2\, dz$, and its centroid is located at $\tilde{z} = z$.

Integrations. Using the third of Eqs. 9–2 with $dm = \rho\, dV$ and integrating with respect to z, noting that $\rho = 200z$, we have

$$\bar{z} = \frac{\int_V \tilde{z}\rho\, dV}{\int_V \rho\, dV} = \frac{\int_0^{1\text{ m}} z(200z)\left[\pi(0.5)^2\, dz\right]}{\int_0^{1\text{ m}} (200z)\pi(0.5)^2\, dz}$$

$$= \frac{\int_0^{1\text{ m}} z^2\, dz}{\int_0^{1\text{ m}} z\, dz} = 0.667\text{ m} \qquad \textit{Ans.}$$

❖ Composite Bodies.

- A composite body consists of a series of connected "simpler" shaped bodies, which may be *rectangular, triangular, semicircular, etc.*
- Such a body can often be *sectioned or divided* into its composite parts and, provided the weight and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body.

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \quad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \quad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W} \quad \ldots\ldots (6)$$

Here

$\bar{x}$, $\bar{y}$, $\bar{z}$ represent the coordinates of the center of gravity G of the composite body.

$\tilde{x}$, $\tilde{y}$, $\tilde{z}$ represent the coordinates of the center of gravity of each composite part of the body.

$\sum W$ is the sum of the weights of all the composite parts of the body, or simply the total weight of the body.

- When the body has a constant density or specific weight , the center of gravity coincides with the centroid of the body. The centroid for composite lines, areas, and volumes can be found using relations analogous to Eqs.6;
- the W 's are replaced by L 's, A 's, and V 's, respectively. Centroids for common shapes of lines, areas, shells, and volumes that often make up a composite body are given in the table.

❖ Procedure for Analysis

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by a line, area, or volume can be determined using the following procedure.

❑ Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole* , or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

❑ Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.

❑ Summations.

- Determine $\bar{x}$, $\bar{y}$, $\bar{z}$ by applying the center of gravity equations, Eqs.6. , or the analogous centroid equations.
- If an object is symmetrical about an axis, the centroid of the object lies on this axis.
- If desired, the calculations can be arranged in tabular form, as indicated in the following examples.

Ex.6. Locate the centroid of the wire shown in Fig. 12 a .

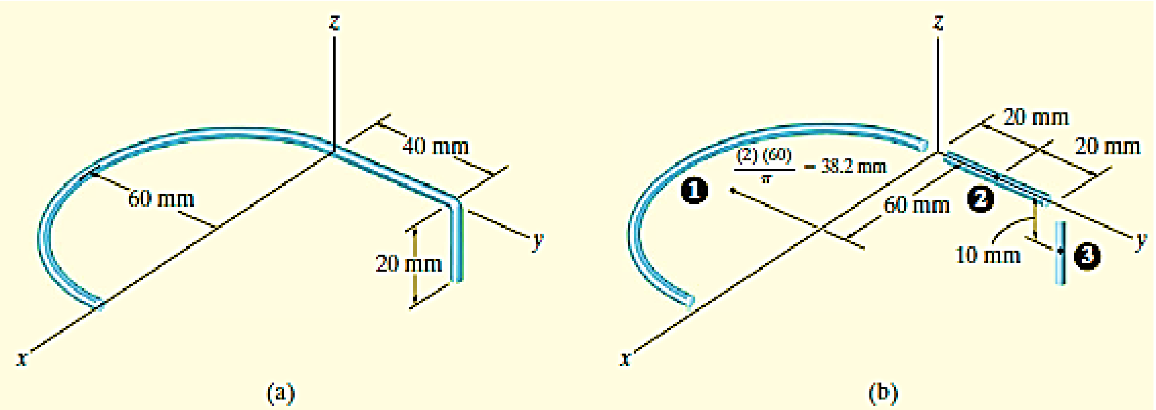


Fig. 12

Sol.

- **Composite Parts.** The wire is divided into three segments as shown in Fig.12b .
- **Moment Arms.** The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment 1 is determined either by integration or by using the table on the inside back cover.
- **Summations.** For convenience, the calculations can be tabulated as follows:

Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	-38.2	0	11 310	-7200	0
2	40	0	20	0	0	800	0
3	20	0	40	-10	0	800	-200
	$\Sigma L = 248.5$				$\Sigma\tilde{x}L = 11\,310$	$\Sigma\tilde{y}L = -5600$	$\Sigma\tilde{z}L = -200$

- Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm}$$

$$\bar{y} = \frac{\Sigma\tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm}$$

$$\bar{z} = \frac{\Sigma\tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm}$$

Ex.7. Locate the centroid of the plate area shown in Fig. 13 a .

Sol.

- Composite Parts. The plate is divided into three segments as shown in Fig. 13 b . Here the area of the small rectangle **3** is considered "negative" since it must be subtracted from the larger one **2** .
- Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of **2** and **3** are negative .
- Summations. Taking the data from Fig. 13 b , the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	-1.5	1.5	-13.5	13.5
3	$-(2)(1) = -2$	-2.5	2	5	-4
	$\Sigma A = 11.5$			$\Sigma\tilde{x}A = -4$	$\Sigma\tilde{y}A = 14$

- Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft}$$

$$\bar{y} = \frac{\Sigma\tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft}$$

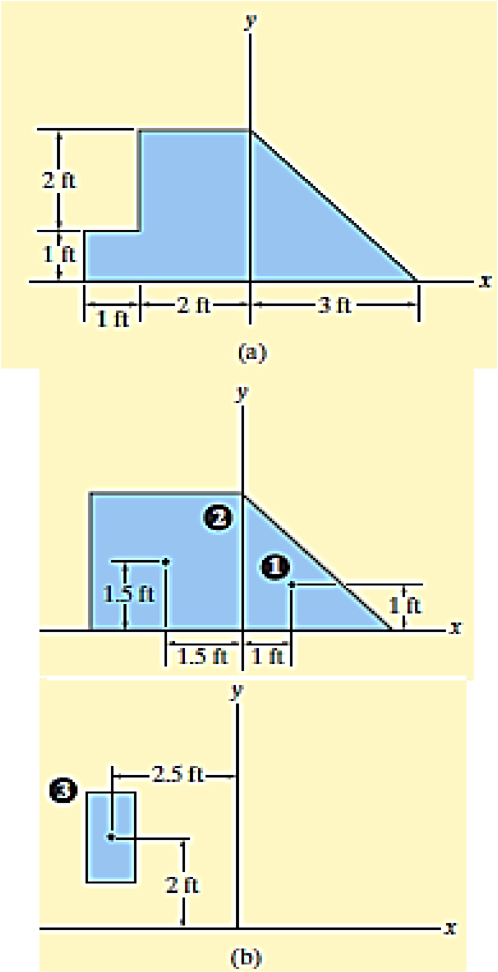
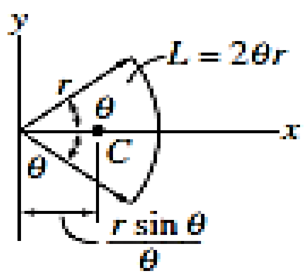


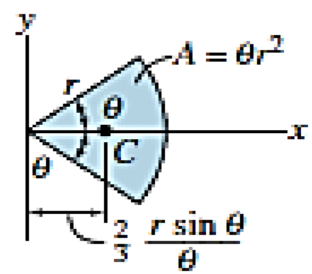
Fig.13.

Geometric Properties of Line and Area Elements

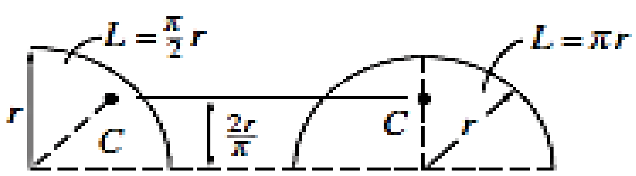
Centroid Location	Centroid Location	A



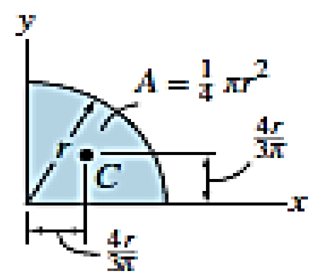
Circular arc segment



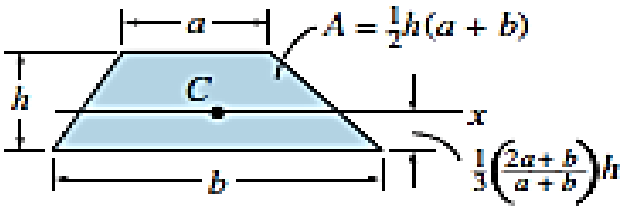
Circular sector area



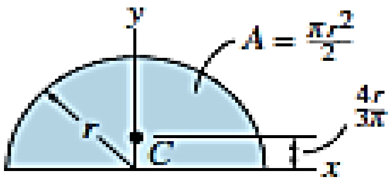
Quarter and semicircle arcs



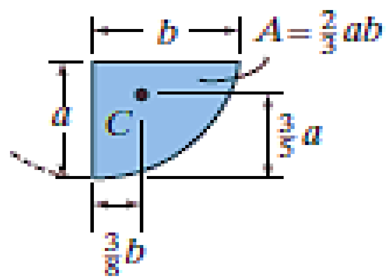
Quarter circle area



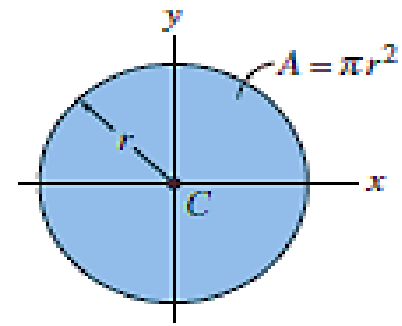
Trapezoidal area



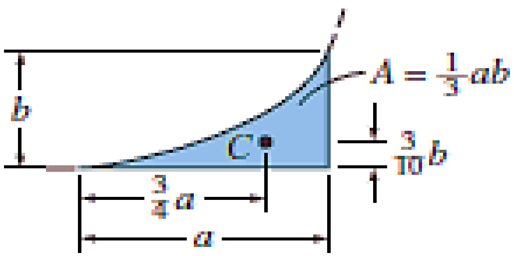
Semicircular area



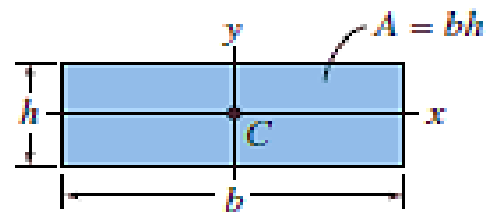
Semiparabolic area



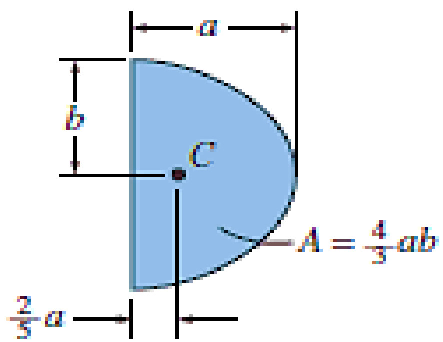
Circular area



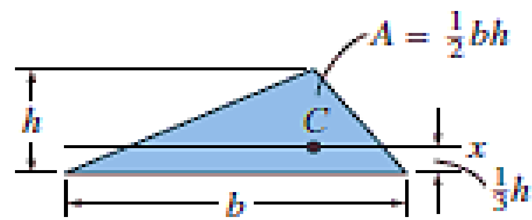
Exparabolic area



Rectangular area



Parabolic area



Triangular area

Moment of Inertia

- **Quantities** called moments of inertia arise repeatedly in analyses of engineering problems.
- Moments of inertia of areas are used in the study of **distributed forces and in calculating deflections of beams.**
- The moment exerted by the pressure on a submerged flat plate can be expressed in terms of the **moment of inertia of the plate's area.**
- In dynamics, mass moments of inertia are used in **calculating the rotational motions of objects.**
- We show how to calculate the moments of inertia of simple areas and objects and then use results **called parallel-axis theorems** to calculate **moments of inertia of more complex areas and objects.**

❖ Definitions

- Consider an area A in the $x - y$ *plane* (Fig.1a). Four moments of inertia of A are defined:

1. Moment of inertia about the x *axis*:

$$\boxed{I_x = \int_A y^2 \, dA,} \quad \ldots\ldots.(1)$$

Where: y is the y *coordinate* of the differential element of *area* dA (Fig.1.b).

- This moment of inertia is sometimes expressed in terms of the radius of gyration about the x *axis*, $\boldsymbol{k_x}$, which is defined by

$$\boxed{I_x = k_x^2 A.} \quad \ldots\ldots..(2)$$

2. Moment of inertia about the y *axis*:

$$\boxed{I_y = \int_A x^2 \, dA,} \quad \ldots\ldots..(3)$$

Where: x is the x *coordinate* of the element dA (Fig.1.b).

- The radius of gyration about the y *axis*, $\boldsymbol{k_y}$, is defined by

$$\boxed{I_y = k_y^2 A.} \quad \ldots\ldots..(4)$$

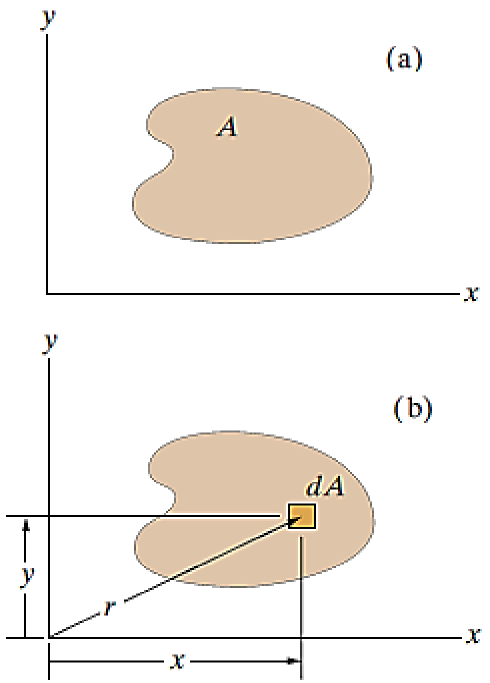


Fig.1.
(a) An area A in the x–y plane.
(b) A differential element of A.

3. Product of inertia:

$$\boxed{I_{xy} = \int_A xy\, dA} \quad \ldots\ldots\ldots (5)$$

4. Polar moment of inertia:

$$\boxed{J_O = \int_A r^2\, dA,} \quad \ldots\ldots\ldots (6)$$

Where: r is the radial distance from the origin of the coordinate system to dA (Fig.1.b).

- The radius of gyration about the origin, $\boldsymbol{k_O}$, is defined by

$$\boxed{J_O = k_O^2\, A} \quad \ldots\ldots\ldots (7)$$

- The polar moment of inertia is equal to the sum of the moments of inertia about the *x and y axes*:

$$\boxed{J_O = \int_A r^2\, dA = \int_A (y^2 + x^2)\, dA = I_x + I_y}$$

- Substituting the expressions for the moments of inertia in terms of the radii of gyration into this equation, we obtain

$$\boxed{k_O^2 = k_x^2 + k_y^2.}$$

- The dimensions of the moments of inertia of an area are $(\boldsymbol{length})^4$ and the radii of gyration have dimensions of $\boldsymbol{length}$.

- Notice that the definitions of the moments of inertia $\boldsymbol{I_x}, \boldsymbol{I_y}$ and $\boldsymbol{J_o}$ and the radii of gyration imply that they have positive values for any area. They cannot be negative or zero.

- If an area A is symmetric about the $x\ axis$, for each element dA with $coordinates\ (x, y\ )$, there is a corresponding element dA with $coordinates (x, -y)$ as shown in Fig.2.

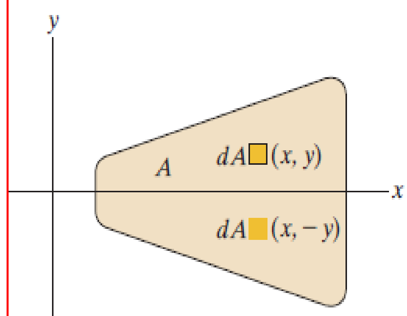


Fig.2.

- The contributions of these two elements to the product of inertia $\boldsymbol{I_{xy}}$ of the area cancel: $xydA + (-xy)dA = 0$ This means that the product of inertia of the area is zero.
- The same kind of argument can be used for an area that is $symmetric\ about\ the\ y\ axis$.

- **If an area is symmetric about either the <u>x axis</u> or the <u>y axis</u>, its product of inertia is zero.**

❖ Parallel-Axis Theorem for an Area

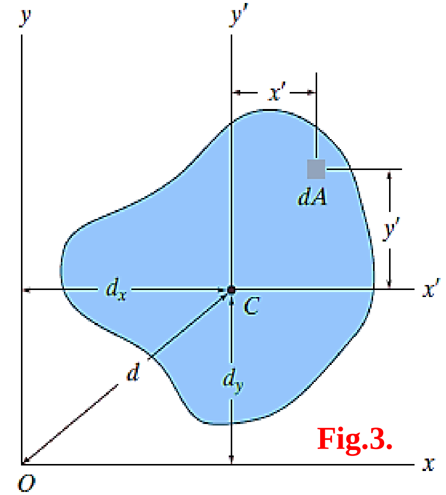


Fig.3.

- The parallel-axis theorem can be used to find the moment of inertia of an area about any axis that is parallel to an axis passing through the centroid and about which the moment of inertia is known.
- To develop this theorem, we will consider finding the moment of inertia of the shaded area shown in Fig.3. about the $x\ axis$.
- To start, we choose a differential element dA located at an arbitrary distance $\boldsymbol{y'}$ from the centroidal $\boldsymbol{x'}axis$.
- If the distance between the parallel $\boldsymbol{x}$ and $\boldsymbol{x'}$ $axes$ is $\boldsymbol{d_y}$, then the moment of inertia of dA about the $x\ axis$ is $dI_x = (y' + d_y)^2 dA$. For the entire area

$$I_x = \int_A (y' + d_y)^2\, dA = \int_A y'^2\, dA + 2d_y \int_A y'\, dA + d_y^2 \int_A dA$$

- The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$.
- The second integral is zero since the $x'axis$ passes through the area's centroid $\boldsymbol{C}$; i.e., $\int y'\, \mathrm{dA} = \overline{y'} \int dA = 0$ since $\overline{y'} = 0$.
- Since the third integral represents the total area A , the final result is therefore

$$I_x = \bar{I}_{x'} + Ad_y^2 \qquad ....(8)$$

- A similar expression can be written for $\boldsymbol{I_y}$; i.e.,

$$I_y = \bar{I}_{y'} + Ad_x^2 \qquad ....(9)$$

- And finally, for the polar moment of inertia, since $\overline{I_C} = \bar{I}_{x'} + \bar{I}_{y'}$ and , $d^2 = d_x^2 + d_y^2$, we have

$$\boxed{J_O = \bar{J}_C + Ad^2}$$ …….(10)

- The form of each of these three equations states that the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes .

❖ Procedure for Analysis

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the arbitrary point (x, y).

❖ Case 1.

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is computed. This situation occurs when the rectangular element shown in Fig. 4*a* is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 4*b*. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

❖ Case 2.

The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is computed; however, Eq.1and 3 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 4*a* is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y.

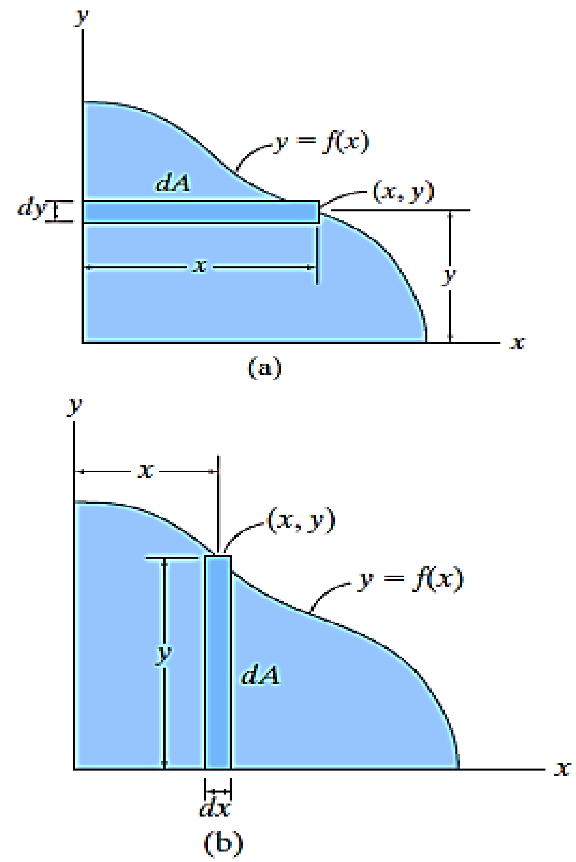


Fig.4.

❖ Moments of Inertia for Composite Areas

- A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the algebraic sum of the moments of inertia of all its parts.

Procedure for Analysis

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the reference axis. For the calculation of $\bar{I}$, use the table on the inside back cover.

Summation.

- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.
- If a composite part has a "hole", its moment of inertia is found by "subtracting" the moment of inertia of the hole from the moment of inertia of the entire part including the hole.

Ex.1. Determine the moment of inertia for the rectangular area shown in *Fig.* 5. with respect to (a) the centroidal x' **axis**, (b) the **axis** x_b passing through the base of the rectangle, and (c) the pole or z' **axis** perpendicular to the $x' - y'$ **plane** and passing through the centroid C .

Sol.

Part (a). The differential element shown in Fig.5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3$$

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2)$$

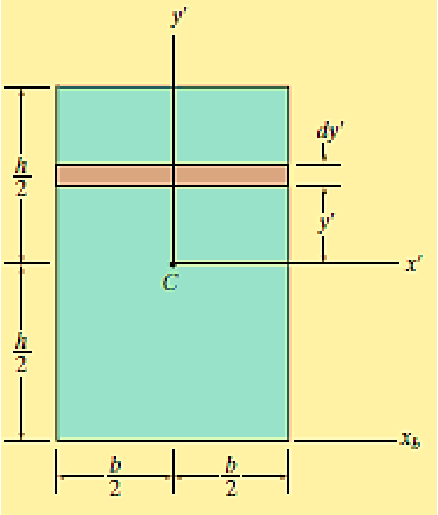


Fig. 5.

Ex.2. Determine the moment of inertia for the shaded area shown in Fig.6. a about the x *axis*.

Sol. (Case 1)

A differential element of area that is *parallel* to the x axis, as shown in Fig. 6a, is chosen for integration. Since this element has a thickness dy and intersects the curve at the *arbitrary point* (x, y), its area is $dA = (100 - x)\,dy$. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

$$I_x = \int_A y^2\,dA = \int_0^{200\text{ mm}} y^2(100 - x)\,dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right) dy = \int_0^{200\text{ mm}} \left(100y^2 - \frac{y^4}{400}\right) dy$$

$$= 107(10^6)\text{ mm}^4$$

Sol. (Case 2)

A differential element *parallel* to the y axis, as shown in Fig. 6b, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the x axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 1. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 6b, $b = dx$ and $h = y$, and thus $d\bar{I}_{x'} = \frac{1}{12}dx\,y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the x axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\,\tilde{y}^2 = \frac{1}{12}dx\,y^3 + y\,dx\left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\,dx$$

(This result can also be concluded from part (b) of Example 1.) Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

$$I_x = \int dI_x = \int_0^{100\text{ mm}} \frac{1}{3}y^3\,dx = \int_0^{100\text{ mm}} \frac{1}{3}(400x)^{3/2}\,dx$$

$$= 107(10^6)\text{ mm}^4$$

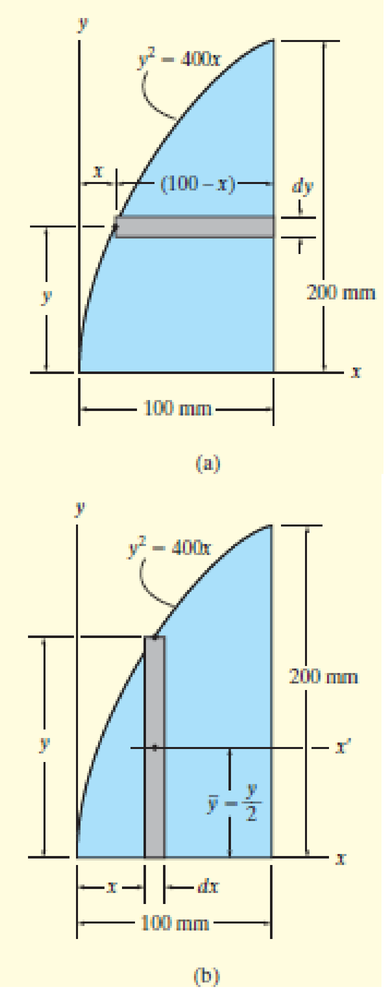
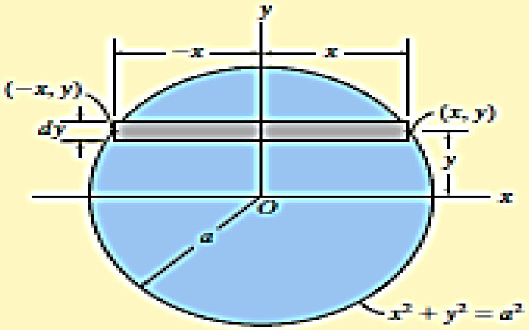


Fig.6.

Ex.3. Determine the moment of inertia with respect to the x axis for the circular area shown in Fig. 7a.



(a)

SOLUTION I (CASE 1)

Using the differential element shown in Fig. 7a, since $dA = 2x\,dy$, we have

$$I_x = \int_A y^2\,dA = \int_A y^2(2x)\,dy$$

$$= \int_{-a}^{a} y^2\left(2\sqrt{a^2 - y^2}\right)dy = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

SOLUTION II (CASE 2)

When the differential element shown in Fig. 7b is chosen, the centroid for the element happens to lie on the x axis, and since $\bar{I}_{x'} = \frac{1}{12}bh^3$ for a rectangle, we have

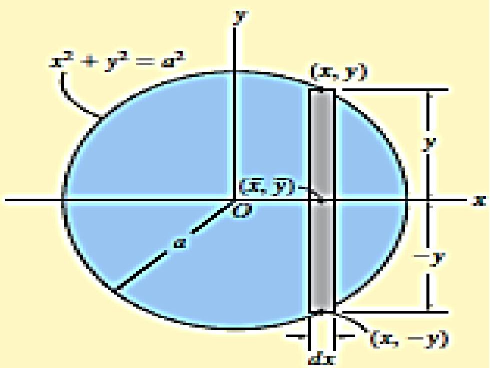
$$dI_x = \frac{1}{12}dx(2y)^3$$

$$= \frac{2}{3}y^3\,dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a} \frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

NOTE: By comparison, Solution I requires much less computation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.



(b)

Fig. 7

Ex.4. Determine the moment of inertia of the area shown in Fig. 8 *a* about the *x axis*.

Sol.

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig.8 *b* . The centroid of each area is located in the figure.

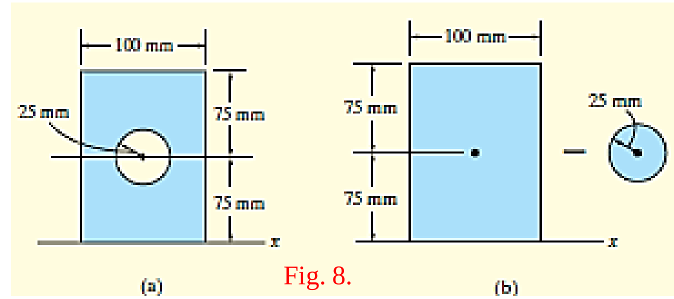


Fig. 8.

Parallel-Axis Theorem. The moments of inertia about the *x axis* are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$, $I_x = \frac{1}{12}bh^3$.

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ \text{mm}^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ \text{mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\ \text{mm}^4$$

Ex.5.

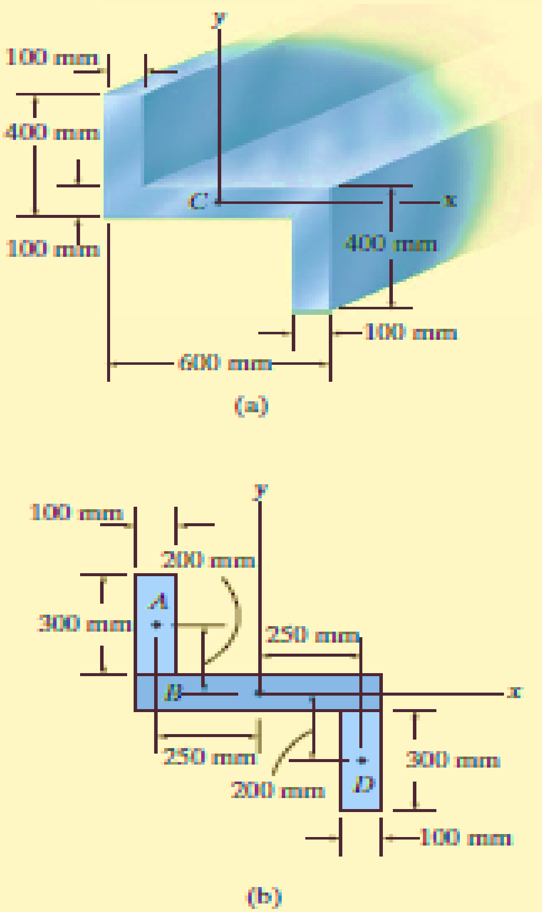


Fig. -9 -

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. -9*a* about the x and y centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas A, B, and D shown in Fig. 9*b*. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles A and D, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\ \text{mm}^4$$
$$I_y = \bar{I}_{y'} + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\ \text{mm}^4$$

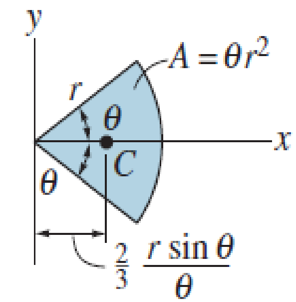
Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\ \text{mm}^4$$
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\ \text{mm}^4$$

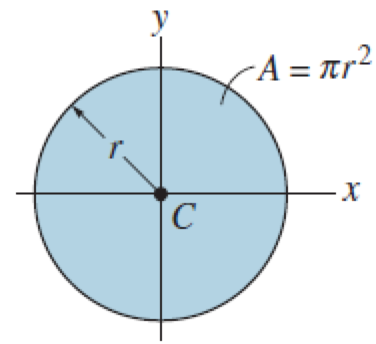
Area Moment of Inertia



$$I_x = \tfrac{1}{4} r^4 (\theta - \tfrac{1}{2}\sin 2)\theta$$

$$I_y = \tfrac{1}{4} r^4 (\theta + \tfrac{1}{2}\sin 2)\theta$$

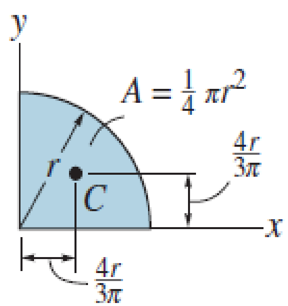
Circular sector area



$$I_x = \tfrac{1}{4}\pi r^4$$

$$I_y = \tfrac{1}{4}\pi r^4$$

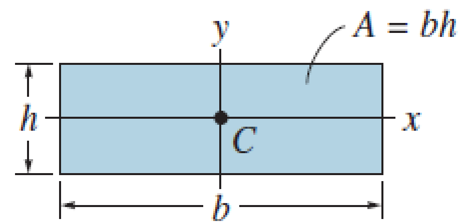
Circular area



$$I_x = \tfrac{1}{16}\pi r^4$$

$$I_y = \tfrac{1}{16}\pi r^4$$

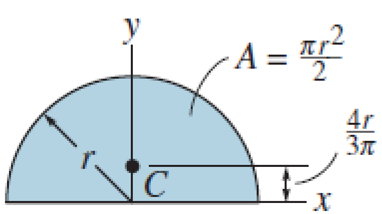
Quarter circle area



$$I_x = \tfrac{1}{12} bh^3$$

$$I_y = \tfrac{1}{12} hb^3$$

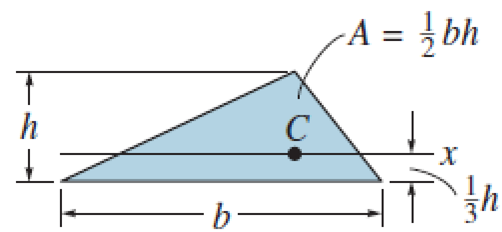
Rectangular area



$$I_x = \tfrac{1}{8}\pi r^4$$

$$I_y = \tfrac{1}{8}\pi r^4$$

Semicircular area



$$I_x = \tfrac{1}{36} bh^3$$

Triangular area

Principles of Dynamics

- **Dynamics.** Which deals with the accelerated motion of a body.

- **Dynamics has two distinct parts:**

1. **Kinematics,** Which treats only the geometric aspects of the motion, i.e. study of motion without reference to the forces which cause motion.

2. **Kinetics**, Kinetics, which is the analysis of the forces causing the motion, i.e. relates to the action of forces on bodies to their resulting motions.

- The principles of dynamics developed when it was possible to make an accurate measurement of time.

- There are many problems in engineering whose solutions require application of the principles of dynamics.
    - Structural design of any vehicle, such as an automobile or airplane, requires consideration of the motion to which it is subjected.

    - Mechanical devices, such as motors, pumps, movable tools, industrial manipulators, and machinery.

    - Predictions of the motions of artificial satellites, projectiles, and spacecraft are based on the theory of dynamics.

    - With further advances in technology, there will be an even greater need for knowing how to apply the principles of this subject.

❑ Rectilinear Kinematics: Continuous Motion

- ❑ In the first part of dynamics, we will start by discussing the kinematics of a particle that moves along a rectilinear or straight-line path.

- ❑ The particle has a mass but negligible size and shape.

- ❑ We will be interested in bodies of finite size, such as rockets, projectiles, or vehicles.

- ❑ Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected.

❖ **Rectilinear Kinematics.** The kinematics of a particle is characterized by specifying, at any given instant, the particle's position, velocity, and acceleration.

- **Position.** The straight-line path of a particle will be defined using a single coordinate $axis\ s$, Fig.1.
- The origin O on the path is a fixed point, and from this point the position $coordinate\ s$ is used to specify the location of the particle at any given instant.
- The magnitude of $\boldsymbol{s}$ is the distance from O to the particle, usually measured in meters (m) or feet (ft), and the sense of $direction$ is defined by the $algebraic\ sign\ on\ s$.
- In this case $s\ is\ positive$ since the coordinate axis is positive to the right of the origin.
- It is $negative$ if the particle is located to the $left\ of\ O$.
- Position is a vector quantity since it has both magnitude and direction.

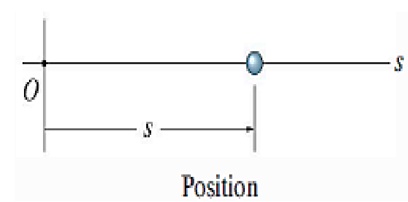


Fig.1.

- **Displacement.** The displacement of the particle is defined as the change in its position. For example, if the particle moves from one point to another, Fig.2., the displacement is

$$\boxed{\Delta s = s' - s} \quad \ldots\ldots.(1)$$

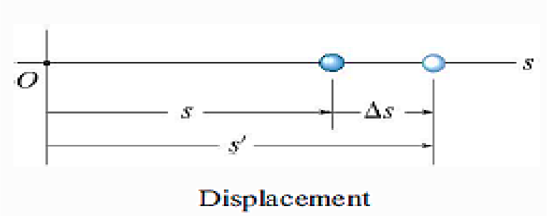


Fig.2.

- In this case **Δs is positive** since the particle's final position is to the *right* of its initial position, i.e., $s' > s$.

- If the final position were to the left of its initial position, Δs would be **negative.**

- The displacement of a particle is also a vector quantity and it should be distinguished from the distance the particle travels.
- The distance traveled is a positive scalar that represents the total length of the path over which the particle travels.

- **Velocity.** If the particle moves through a displacement Δs during the time interval Δt, the *average velocity* of the particle during this time interval is

$$\boxed{v_{avg} = \frac{\Delta s}{\Delta t}} \quad ....(2)$$

- If we take smaller and smaller values of Δt, the magnitude of Δs becomes smaller and smaller. Consequently, the instantaneous velocity is a vector defined as

$$\boxed{v = \lim_{\Delta t \to 0} (\Delta s / \Delta t)} \quad ....(3)$$

- Or

$$(\overset{+}{\rightarrow}) \quad \boxed{v = \frac{ds}{dt}} \quad .....(4)$$

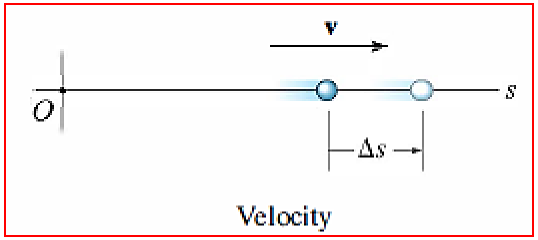


Fig.3.

- Since Δt or dt is always positive, the sign used to define the sense of the velocity is the same as that of Δs or ds.
- For example, if the particle is moving to the *right*, Fig.3, the *velocity is positive*; whereas if it is moving to the *left*, the *velocity is negative*.
- The *magnitude* of the velocity is known as the *speed*, and it is generally expressed in units of m/s or ft/s.
- The *average speed* is always a *positive scalar* and is defined as the *total distance traveled by a particle*, $\boldsymbol{S_T}$, divided by the elapsed time $\boldsymbol{\Delta t}$; i.e.,

$$\boxed{(v_{sp})_{avg} = \frac{s_T}{\Delta t}} \quad .....(5)$$

- Particle in **Fig.4** travels along the path of length $\boldsymbol{S_T}$ in time $\boldsymbol{\Delta t}$, so its average speed is $(v_{sp})_{avg} = s_T/\Delta t$, but its average velocity is $v_{avg} = -\, s_T/\Delta t$.

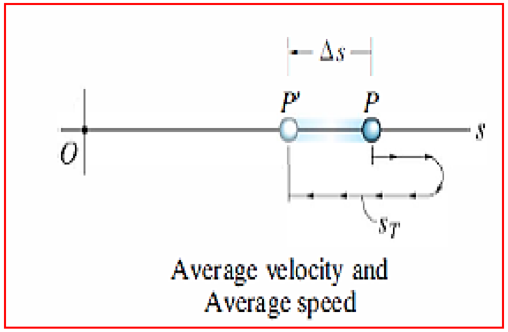


Fig.4

- **Acceleration.** Provided the velocity of the particle is known at two points, the *average acceleration* of the particle during the *time interval* Δt is defined as

$$\boxed{a_{avg} = \frac{\Delta v}{\Delta t}} \quad .....(6)$$

- Δv represents the difference in the velocity during the time interval Δt, i.e., $\Delta v = v' - v$, Fig.5.

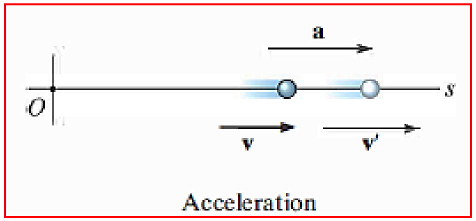


Fig.5.

- The *instantaneous acceleration* at time t is a vector that is found by taking smaller and smaller values of Δt and corresponding smaller and smaller values of Δv, so that $a = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t}\right)$, or

$$(\overset{+}{\rightarrow}) \quad \boxed{a = \frac{dv}{dt}} \quad .....(7)$$

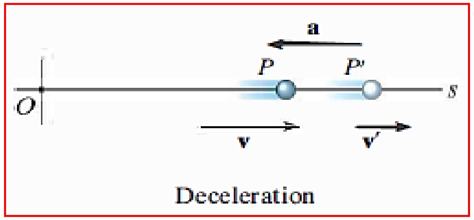


Fig.6.

- Substituting Eq.4 into this result, we can also write $(\overset{+}{\rightarrow}) \quad \boxed{a = \frac{d^2s}{dt^2}} \quad .....(8)$
- Both the average and instantaneous acceleration can be either *positive or negative*.
- when the particle is slowing down, or its speed is decreasing, the particle is said to be *decelerating*. In this case, v' in Fig.6 is less than v and so $\Delta v = v' - v$ will be *negative*.
- a will also be *negative*, and therefore it will act to the *left*, in the *opposite sense* to v. Also, note that when the velocity is constant, the *acceleration* is *zero* since $\Delta v = v - v = 0$.
- Units commonly used to express the magnitude of acceleration are $^{m}/_{s^2}$ or ft/s^2.

- The important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential dt between Eqs. 4 and 7, which gives

$$(\overset{+}{\rightarrow}) \qquad \boxed{a\, ds = v\, dv} \qquad \ldots..(9)$$

- **Constant Acceleration**, $a = a_c$.When the acceleration is constant, each of the three kinematic equations $a_c = \frac{dv}{dt}, v = \frac{ds}{dt}$, and $a_c ds = v\, dv$ can be integrated to obtain formulas that relate $a_c, v, s, and\ t$.

- **Velocity as a Function of Time**. Integrate $a_c = dv/dt$, assuming that initially $v = v_0$ when $t = 0$.

$$\int_{v_0}^{v} dv = \int_0^t a_c\, dt$$

$$(\overset{+}{\rightarrow}) \qquad \boxed{\begin{array}{c} v = v_0 + a_c t \\ \text{Constant Acceleration} \end{array}} \qquad \ldots..(10)$$

- **Position as a Function of Time.** Integrate $v = ds/dt = v_0 + a_c t$, assuming that initially $s = s_0$when $t = 0$.

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\, dt$$

$$(\overset{+}{\rightarrow}) \qquad \boxed{\begin{array}{c} s = s_0 + v_0 t + \frac{1}{2} a_c t^2 \\ \text{Constant Acceleration} \end{array}} \qquad \ldots..(11)$$

- **Velocity as a Function of Position**. Either solve for t in Eq. 10 and substitute into Eq. 11, or integrate $v\,dv = a_c ds$, assuming that initially $v = v_0$ at $s = s_0$.

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a_c\,ds$$

$$(\overset{+}{\rightarrow}) \qquad \boxed{\begin{array}{c} v^2 = v_0^2 + 2a_c(s - s_0) \\ \text{Constant Acceleration} \end{array}} \qquad \ldots..(12)$$

- The algebraic signs of $s_0, v_0, and\ a_c$, used in the above three equations, are determined from the positive direction of the s axis as indicated by the arrow written at the left of each equation.

- Remember that these equations are useful only when the acceleration is constant and when $t = 0, s = s_0, v = v_0$.

- A typical example of constant accelerated motion occurs when a body falls freely toward the earth.

- If air resistance is neglected and the distance of fall is short, then the downward acceleration of the body when it is close to the earth is constant and approximately 9.81 m/s^2 or 32.2 ft/s^2.

❖ Procedure for Analysis

- **Coordinate System.**

- Establish a position coordinate s along the path and specify its *fixed origin* and positive direction.
- Since motion is along a straight line, the vector quantities position, velocity, and acceleration can be represented as algebraic scalars. For analytical work the sense of $s, v, and\ a$ is then defined by their algebraic signs.
- The positive sense for each of these scalars can be indicated by an arrow shown alongside each kinematic equation as it is applied.

- **Kinematic Equations.**

- If a relation is known between any two of the four variables a, v, sand t, then a third variable can be obtained by using one of the kinematic equations, $a = dv/dt, v = ds/dt\ or\ a\,ds = v\,dv$, since each equation relates all three variables.
- Whenever integration is performed, it is important that the position and velocity be known at a given instant in order to evaluate either the constant of integration if an indefinite integral is used, or the limits of integration if a definite integral is used.
- Remember that Eqs. 10 through 12 have only limited use. These equations apply only when the acceleration is constant and the initial conditions are $s = s_0$ and $v = v_0$ when $t = 0$.

Ex.1. The car in Fig. 7 moves in a straight line such that for a short time its velocity is defined by $v = (3t^2 + 2t)\ ft/s$, where t is in *seconds*. Determine its position and acceleration when $t = 3\ s$. When $t = 0, s = 0$.

Sol.

Fig.7.

- Coordinate System. The position coordinate extends from the *fixed origin O* to the car, positive to the right.

- Position. Since $v = f(t)$, the car's position can be determined from $v = ds/dt$, since this equation relates $v, s, and\ t$. Noting that $s = 0$ when $t = 0$, we have

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} = (3t^2 + 2t)$$

$$\int_0^s ds = \int_0^t (3t^2 + 2t)dt$$

$$s\Big|_0^s = t^3 + t^2\Big|_0^t$$

$$s = t^3 + t^2$$

When $t = 3$ s,

$$s = (3)^3 + (3)^2 = 36\text{ ft}$$

The same result can be obtained by evaluating a constant of integration C rather than using definite limits on the integral. For example, integrating $ds = (3t^2 + 2t)dt$ yields $s = t^3 + t^2 + C$. Using the condition that at $t = 0, s = 0$, then $C = 0$.

- Acceleration. Since $v = f(t)$, the acceleration is determined from $a = dv/dt$, since this equation relates $a, v, and\ t$.

$$(\overset{+}{\rightarrow}) \qquad a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 2t)$$

$$= 6t + 2$$

When $t = 3$ s,

$$a = 6(3) + 2 = 20\text{ ft/s}^2 \rightarrow$$

- NOTE: The formulas for constant acceleration cannot be used to solve this problem, because the acceleration is a function of time.

Ex.2. During a test a rocket travels upward at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height s_B reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of 9.81 m/s^2 due to gravity. Neglect the effect of air resistance.

Sol.

❖ Coordinate System. The origin O for the position *coordinate s* is taken at ground level with positive upward, Fig. 8.

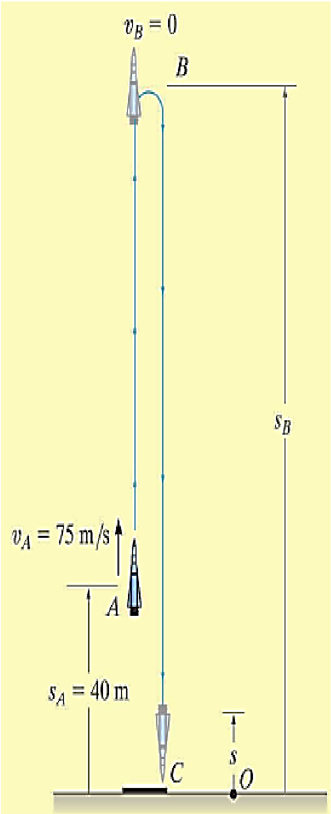


Fig.8.

❖ Maximum Height.

- Since the rocket is traveling upward, $v_A = +75 m/s$ when $t = 0$.
- At the maximum height $s = s_B$ the velocity $v_B = 0$.
- For the entire motion, the acceleration is $a_c = -9.81\ m/s^2$ (negative since it acts in the opposite sense to positive velocity or positive displacement). Since a_c is constant the rocket's position may be related to its velocity at the two points A and B on the path by using Eq. 12, namely

$$(+\uparrow) \qquad v_B^2 = v_A^2 + 2a_c(s_B - s_A)$$
$$0 = (75\ \text{m/s})^2 + 2(-9.81\ \text{m/s}^2)(s_B - 40\ \text{m})$$
$$s_B = 327\ \text{m}$$

❖ Velocity.

- To obtain the velocity of the rocket just before it hits the ground, we can apply Eq. 12 between points B and C, Fig8.

$$(+\uparrow) \qquad v_C^2 = v_B^2 + 2a_c(s_C - s_B)$$
$$= 0 + 2(-9.81\ \text{m/s}^2)(0 - 327\ \text{m})$$
$$v_C = -80.1\ \text{m/s} = 80.1\ \text{m/s} \downarrow$$

- The negative root was chosen since the rocket is moving downward. Similarly, Eq. 1 may also be applied between points A and C, i.e.,

$$(+\uparrow) \qquad v_C^2 = v_A^2 + 2a_c(s_C - s_A)$$
$$= (75\ \text{m/s})^2 + 2(-9.81\ \text{m/s}^2)(0 - 40\ \text{m})$$
$$v_C = -80.1\ \text{m/s} = 80.1\ \text{m/s} \downarrow$$

It should be realized that the rocket is subjected to a deceleration from A to B of 9.81 m/s^2, and then from B to C it is accelerated at this rate. Furthermore, even though the rocket momentarily comes to rest at B ($v_B = 0$) the acceleration at B is still 9.81 m/s^2 downward.

Ex.3. A small projectile is fired vertically *downward* into a fluid medium with an initial velocity of $60\ m/s$. Due to the drag resistance of the fluid the projectile experiences a deceleration of $a = (-0.4v^3)\ m/s^2$, where v is in m/s. Determine the projectile's velocity and position $4\ s$ after it is fired.

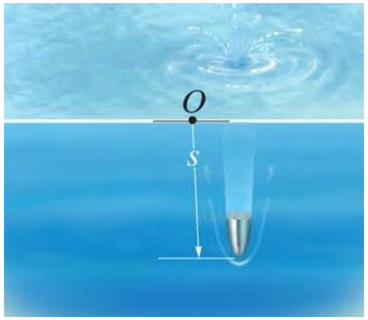


Fig. 9.

Sol.

❖ Coordinate System. Since the motion is downward, the position coordinate is positive downward, with origin located at O, Fig.9.

❖ Velocity. Here $a = f(v)$ and so we must determine the velocity as a function of time using $a = dv/dt$, since this equation relates $v, a, and\ t$. Separating the variables and integrating, with $v_o = 60\ m/s$ when $t = 0$, yields

$(+\downarrow)$

$$a = \frac{dv}{dt} = -0.4v^3$$

$$\int_{60\ \mathrm{m/s}}^{v} \frac{dv}{-0.4v^3} = \int_0^t dt$$

$$\frac{1}{-0.4}\left(\frac{1}{-2}\right)\frac{1}{v^2}\Bigg|_{60}^{v} = t - 0$$

$$\frac{1}{0.8}\left[\frac{1}{v^2} - \frac{1}{(60)^2}\right] = t$$

$$v = \left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}\right\} \mathrm{m/s}$$

Here the positive root is taken, since the projectile will continue to move downward. When $t = 4\ s$,

$$v = 0.559\ \mathrm{m/s}\downarrow$$

Position. Knowing $v = f(t)$, we can obtain the projectile's position from $v = ds/dt$, since this equation relates $s, v, and\ t$. Using the initial condition $s = 0$, when $t = 0$, we have

$(+\downarrow)$

$$v = \frac{ds}{dt} = \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2}$$

$$\int_0^s ds = \int_0^t \left[\frac{1}{(60)^2} + 0.8t\right]^{-1/2} dt$$

$$s = \frac{2}{0.8}\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2}\Bigg|_0^t$$

$$s = \frac{1}{0.4}\left\{\left[\frac{1}{(60)^2} + 0.8t\right]^{1/2} - \frac{1}{60}\right\} \mathrm{m}$$

When $t = 4\ s$

$$s = 4.43\ m$$

Curvilinear motion

- **Curvilinear motion** occurs when a particle moves along a curved path.
- Since this path is often described in three dimensions, vector analysis will be used to formulate the particle's position, velocity, and acceleration.

❖ Position.

- Consider a particle located at a point on a space curve defined by the path function $s(t)$ as shown in Fig.1.
- The position of the particle, measured from a fixed point ***O***, will be designated by the *position vector* $r = r(t)$.
- The magnitude and direction of this vector will change as the particle moves along the curve.

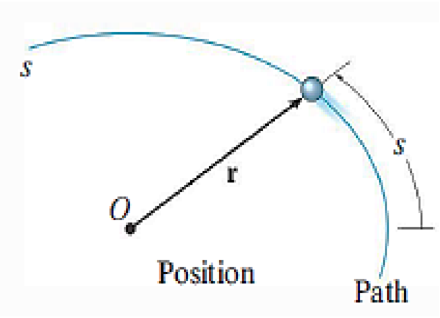


Fig.1.

❖ Displacement.

- Suppose that during a small time interval Δt the particle moves a distance Δs along the curve to a new position, defined by $r' = r + \Delta r$, as shown in Fig.2.
- The displacement Δr represents the change in the particle's position and is determined by vector subtraction; i.e., $\Delta r = r' - r$.

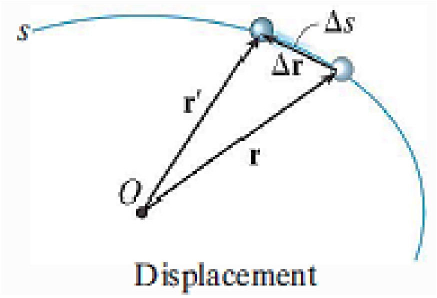


Fig.2.

❖ Velocity.

- During the time Δt, the average velocity of the particle is $v_{avg} = \frac{\Delta r}{\Delta t}$
- The instantaneous velocity is determined from this equation by letting $\Delta t \rightarrow 0$.
- The direction of Δr approaches the tangent to the curve.
- Hence, $v = \lim_{\Delta t \to 0}\left(\frac{\Delta r}{\Delta t}\right) \xrightarrow{or} v = \frac{dr}{dt} \ldots\ldots..(1)$
- Since dr will be tangent to the curve, the direction of v is also tangent to Fig.3.

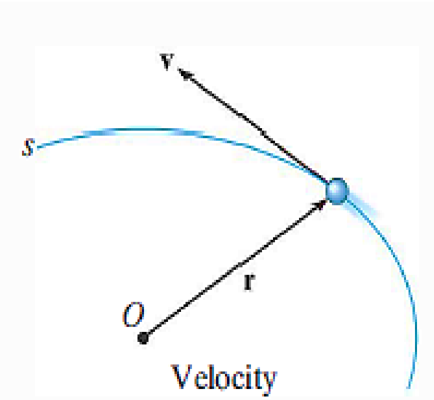


Fig.3.

- The magnitude of v, which is called the speed, is obtained by realizing that the length of the straight line segment Δr in Fig.2. approaches the arc length Δs as $\Delta t \rightarrow 0$, we have ,

$$v = \lim_{\Delta t \to 0}\left(\frac{\Delta r}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta s}{\Delta t}\right) \xrightarrow{or} v = \frac{ds}{dt} \ldots\ldots\ldots.(2)$$

- Thus, the *speed* can be obtained by differentiating the path function *s* with respect to time.

❖ Acceleration.

- If the particle has a velocity v at time t and a velocity $v' = v + \Delta v$ at $t + \Delta t$, as shown in Fig.4. , then the *average acceleration* of the particle during the time interval Δt is $a_{avg} = \frac{\Delta v}{\Delta t}$
- where $\Delta v = v' - v$
- To study this time rate of change, the two velocity vectors in Fig. 4 are plotted in Fig. 5 such that their tails are located at the fixed point O' and their arrowheads touch points on a curve.
- This curve is called a hodograph, and when constructed, it describes the locus of points for the arrowhead of the velocity vector in the same manner as the path s describes the locus of points for the arrowhead of the position vector, Fig.1.

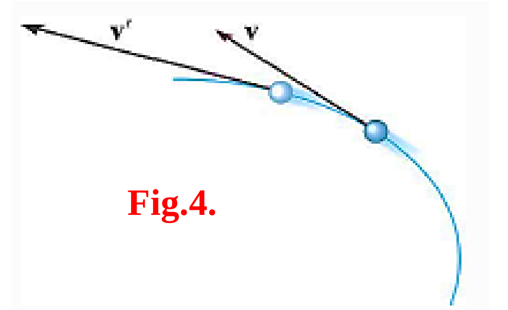


Fig.4.

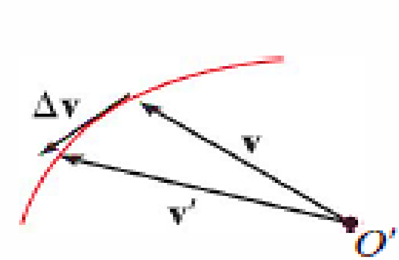


Fig.5.

- To obtain the *instantaneous acceleration*, let $\Delta t \to 0$ in the above equation. In the limit Δv will approach the tangent to the hodograph, and so
- $a = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t}\right) \xrightarrow{\text{or}} a = \frac{dv}{dt} \ldots\ldots\ldots (3)$
- By Subs. Eq. 1. into this result, we can also write $a = \frac{d^2 r}{dt^2}$

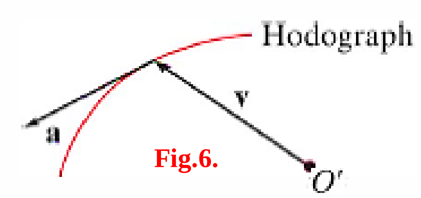


Fig.6.

- By definition of the derivative, a acts tangent to the hodograph, Fig. 6, and, in general it is not tangent to the path of motion, Fig. 7.
- In summary, v is always tangent to the path and a is always tangent to the hodograph.

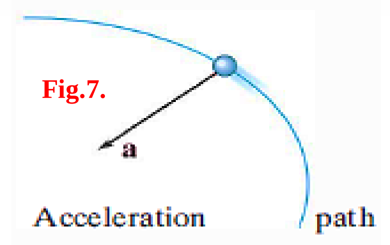


Fig.7.

Curvilinear Motion: Rectangular Components

- Occasionally the motion of a particle can best be described along a path that can be expressed in terms of its x, y, z coordinates.

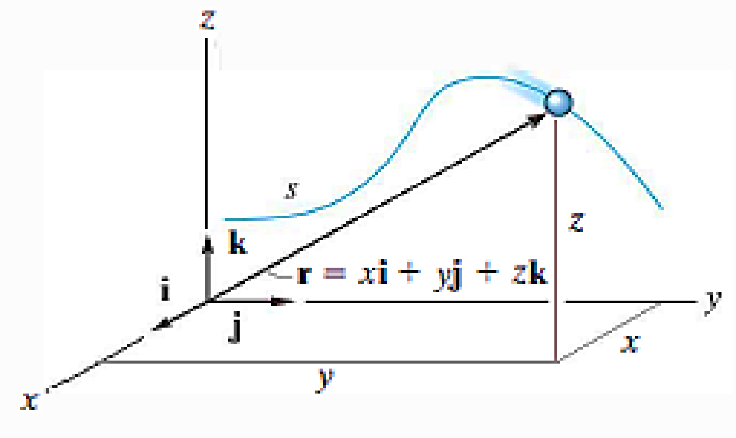


Fig. 8. Position

❖ Position.

- If the particle is at point (x, y, z) on the curved path s shown in Fig. 8, then its location is defined by the *position vector*

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \ldots\ldots(4)$$

- When the particle moves, the x, y, z components of r will be functions of time; i.e., $x = x(t), y = y(t), z = z(t)$, so that $r = r(t)$.

- At any instant the magnitude of r is defined as $r = \sqrt{\boldsymbol{x^2 + y^2 + z^2}}$

- And the direction of r is specified by the unit vector $\boldsymbol{u_r} = \dfrac{\boldsymbol{r}}{r}$

❖ Velocity.

- The first time derivative of $\boldsymbol{r}$ yields the velocity of the particle. Hence,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$

- It is necessary to account for changes in *both* the magnitude and direction of each of the vector's components.
- The derivative of the **i** component of **r** is $\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt}$
- The second term on the right side is zero, provided the x, y, z reference frame is *fixed*, and therefore the direction and the magnitude of **i** does not change with time.
- Differentiation of the **j** and **k** components may be carried out in a similar manner, which yields the final result,

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}} \quad \ldots\ldots..(5)$$

- where

$$\boxed{v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z}} \quad \ldots\ldots..(6)$$

- The "dot" notation $\dot{x}, \dot{y}, \dot{z}$ represents the first time derivatives of $x = x(t), y = y(t), z = z(t)$, respectively.
- The velocity has a magnitude that is found from $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$
- And a direction that is specified by the unit vector $\boldsymbol{u}_v = \frac{\mathbf{v}}{v}$.
- this direction is *always tangent to the path*, as shown in Fig.9.

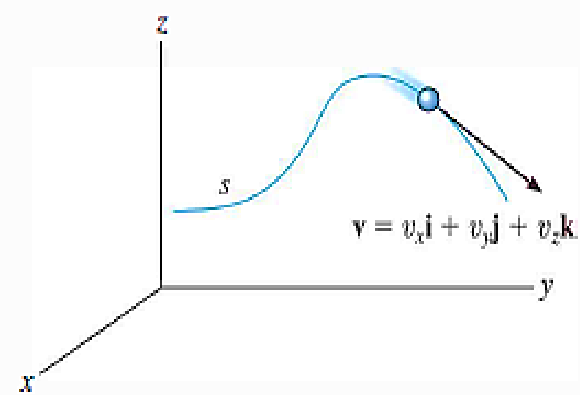


Fig.9. Velocity

❖ Acceleration.

- The acceleration of the particle is obtained by taking the first time derivative of Eq. 5. (or the second time derivative of Eq. 4.). We have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad \text{.........(7)}$$

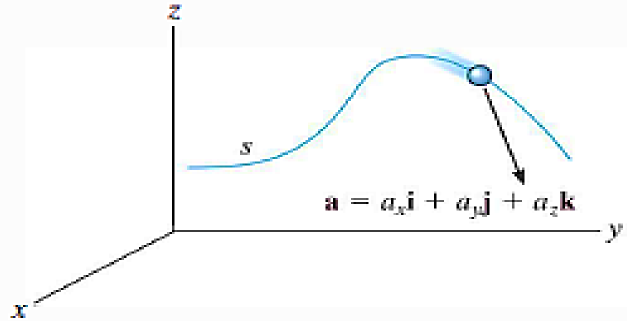


Fig. 10. Acceleration

- Where

$$\begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{v}_z = \ddot{z} \end{aligned} \quad \text{.........(8)}$$

- Here a_x , a_y, a_z represent, respectively, the first time derivatives of $v_x = v_x(\mathrm{t})$, $v_y = v_y(\mathrm{t})$, $v_z = v_z(\mathrm{t})$.

- Or the second time derivatives of the functions $x = x(t), y = y(t), z = z(t)$.

- The acceleration has a magnitude $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

- and a direction specified by the unit vector $\boldsymbol{u}_a = \dfrac{\mathbf{a}}{a}$.

- Since **a** represents the time rate of *change in both the magnitude and direction* of the velocity, in general **a** will *not* be tangent to the path, as shown in Fig. 10.

Motion of a Projectile

- The free-flight motion of a projectile is often studied in terms of its rectangular components.

- To illustrate the kinematic analysis, consider a projectile launched at point (x_o, y_o), with an initial velocity of $\mathbf{v_o}$, having components $(\mathbf{v_o})_x$ and $(\mathbf{v_o})_y$, Fig. 11.

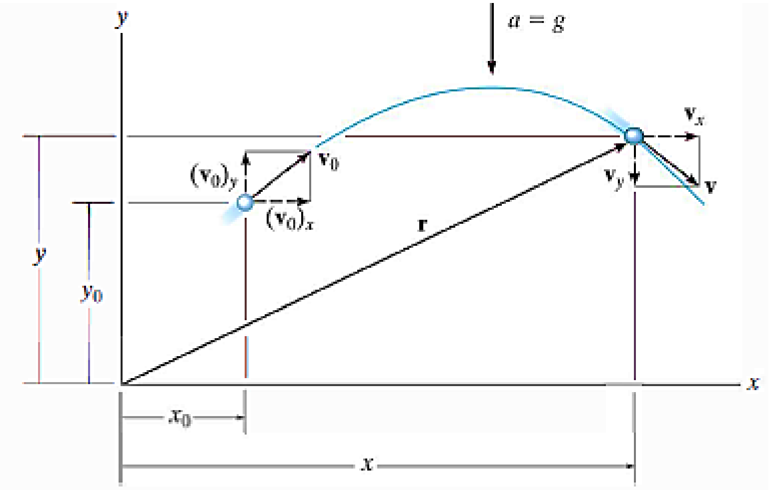


Fig. 11.

- Air resistance is neglected, the only force acting on the projectile is its weight, which causes the projectile to have a *constant downward acceleration* of approximately $a_c = g = 9.81\frac{m}{s^2}$ or $g = 32.2\frac{ft}{s^2}$.

❖ Horizontal Motion.

- Since $a_x = 0$, application of the constant acceleration equations, yields

$$
\begin{array}{lll}
(\overset{+}{\rightarrow}) & v = v_0 + a_c t; & v_x = (v_0)_x \\
(\overset{+}{\rightarrow}) & x = x_0 + v_0 t + \frac{1}{2} a_c t^2; & x = x_0 + (v_0)_x t \\
(\overset{+}{\rightarrow}) & v^2 = v_0^2 + 2a_c(x - x_0); & v_x = (v_0)_x
\end{array}
$$

- The First and last equations indicate that the *horizontal component of velocity always remains constant during the motion.*

❖ Vertical Motion.

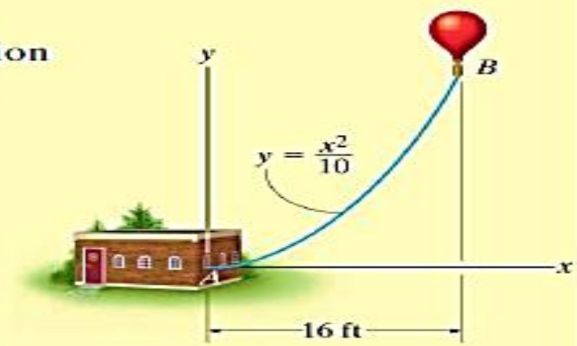
- Since the positive *y axis* is directed upward, then $a_y = -g$, we get

$$
\begin{array}{lll}
(+\uparrow) & v = v_0 + a_c t; & v_y = (v_0)_y - gt \\
(+\uparrow) & y = y_0 + v_0 t + \frac{1}{2} a_c t^2; & y = y_0 + (v_0)_y t - \frac{1}{2} g t^2 \\
(+\uparrow) & v^2 = v_0^2 + 2a_c(y - y_0); & v_y^2 = (v_0)_y^2 - 2g(y - y_0)
\end{array}
$$

- Recall that the last equation can be formulated on the basis of eliminating the time *t* from the first two equations, and therefore *only two of the above three equations are independent of one another.*

EXAMPLE 1

At any instant the horizontal position of the weather balloon in Fig. 12 is defined by $x = (8t)$ ft, where t is in seconds. If the equation of the path is $y = x^2/10$, determine the magnitude and direction of the velocity and the acceleration when $t = 2$ s.



(a)

SOLUTION

Velocity. The velocity component in the x direction is

$$v_x = \dot{x} = \frac{d}{dt}(8t) = 8 \text{ ft/s} \rightarrow$$

To find the relationship between the velocity components we will use the chain rule of calculus. (See Appendix A for a full explanation.)

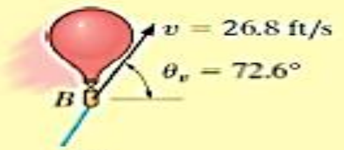
$$v_y = \dot{y} = \frac{d}{dt}(x^2/10) = 2x\dot{x}/10 = 2(16)(8)/10 = 25.6 \text{ ft/s} \uparrow$$

When $t = 2$ s, the magnitude of velocity is therefore

$$v = \sqrt{(8 \text{ ft/s})^2 + (25.6 \text{ ft/s})^2} = 26.8 \text{ ft/s}$$

The direction is tangent to the path, Fig. 12b, where

$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{25.6}{8} = 72.6^\circ$$



(b)

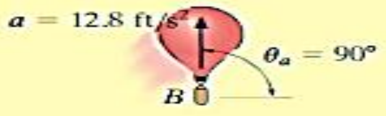
Acceleration. The relationship between the acceleration components is determined using the chain rule. (See Appendix C.) We have

$$a_x = \dot{v}_x = \frac{d}{dt}(8) = 0$$

$$a_y = \dot{v}_y = \frac{d}{dt}(2x\dot{x}/10) = 2(\dot{x})\dot{x}/10 + 2x(\ddot{x})/10$$
$$= 2(8)^2/10 + 2(16)(0)/10 = 12.8 \text{ ft/s}^2 \uparrow$$

Thus,

$$a = \sqrt{(0)^2 + (12.8)^2} = 12.8 \text{ ft/s}^2$$



(c)

The direction of **a**, as shown in Fig. 12c, is

$$\theta_a = \tan^{-1}\frac{12.8}{0} = 90^\circ$$

Fig. 12

EXAMPLE 2.

For a short time, the path of the plane in Fig. 13a is described by $y = (0.001x^2)$ m. If the plane is rising with a constant velocity of 10 m/s, determine the magnitudes of the velocity and acceleration of the plane when it is at $y = 100$ m.

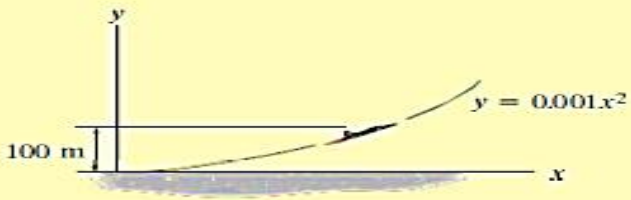
SOLUTION

When $y = 100$ m, then $100 = 0.001x^2$ or $x = 316.2$ m. Also, since $v_y = 10$ m/s, then

$$y = v_y t; \qquad 100\text{ m} = (10\text{ m/s})\,t \qquad t = 10\text{ s}$$

Velocity. Using the chain rule (see Appendix C) to find the relationship between the velocity components, we have

$$v_y = \dot{y} = \frac{d}{dt}(0.001x^2) = (0.002x)\dot{x} = 0.002xv_x \tag{1}$$



(a)

Thus

$$10\text{ m/s} = 0.002(316.2\text{ m})(v_x)$$
$$v_x = 15.81\text{ m/s}$$

The magnitude of the velocity is therefore

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.81\text{ m/s})^2 + (10\text{ m/s})^2} = 18.7\text{ m/s}$$

Acceleration. Using the chain rule, the time derivative of Eq. (1) gives the relation between the acceleration components.

$$a_y = \dot{v}_y = 0.002\dot{x}v_x + 0.002x\dot{v}_x = 0.002(v_x^2 + xa_x)$$

When $x = 316.2$ m, $v_x = 15.81$ m/s, $\dot{v}_y = a_y = 0$,

$$0 = 0.002((15.81\text{ m/s})^2 + 316.2\text{ m}(a_x))$$
$$a_x = -0.791\text{ m/s}^2$$

(b)

Fig. 13

The magnitude of the plane's acceleration is therefore

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791\text{ m/s}^2)^2 + (0\text{ m/s}^2)^2}$$
$$= 0.791\text{ m/s}^2$$

These results are shown in Fig. 13b.

EXAMPLE 12.11

A sack slides off the ramp, shown in Fig. 12–21, with a horizontal velocity of 12 m/s. If the height of the ramp is 6 m from the floor, determine the time needed for the sack to strike the floor and the range R where sacks begin to pile up.

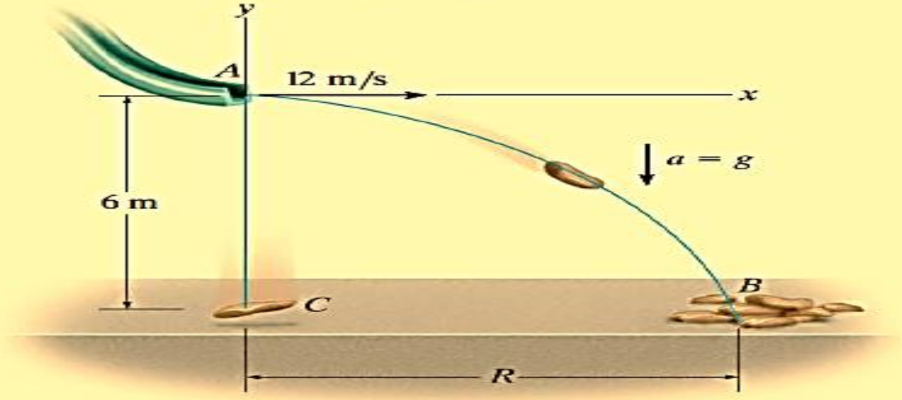


Fig. 12–21

SOLUTION

Coordinate System. The origin of coordinates is established at the beginning of the path, point A, Fig. 12–21. The initial velocity of a sack has components $(v_A)_x = 12$ m/s and $(v_A)_y = 0$. Also, between points A and B the acceleration is $a_y = -9.81 \text{ m/s}^2$. Since $(v_B)_x = (v_A)_x = 12$ m/s, the three unknowns are $(v_B)_y$, R, and the time of flight t_{AB}. Here we do not need to determine $(v_B)_y$.

Vertical Motion. The vertical distance from A to B is known, and therefore we can obtain a direct solution for t_{AB} by using the equation

$(+\uparrow)$
$$y_B = y_A + (v_A)_y t_{AB} + \tfrac{1}{2} a_c t_{AB}^2$$
$$-6 \text{ m} = 0 + 0 + \tfrac{1}{2}(-9.81 \text{ m/s}^2) t_{AB}^2$$
$$t_{AB} = 1.11 \text{ s} \qquad \textit{Ans.}$$

Horizontal Motion. Since t_{AB} has been calculated, R is determined as follows:

$(\xrightarrow{+})$
$$x_B = x_A + (v_A)_x t_{AB}$$
$$R = 0 + 12 \text{ m/s } (1.11 \text{ s})$$
$$R = 13.3 \text{ m} \qquad \textit{Ans.}$$

NOTE: The calculation for t_{AB} also indicates that if a sack were released *from rest* at A, it would take the same amount of time to strike the floor at C, Fig. 12–21.

EXAMPLE 12.12

The chipping machine is designed to eject wood chips at $v_O = 25$ ft/s as shown in Fig. 12–22. If the tube is oriented at 30° from the horizontal, determine how high, h, the chips strike the pile if at this instant they land on the pile 20 ft from the tube.

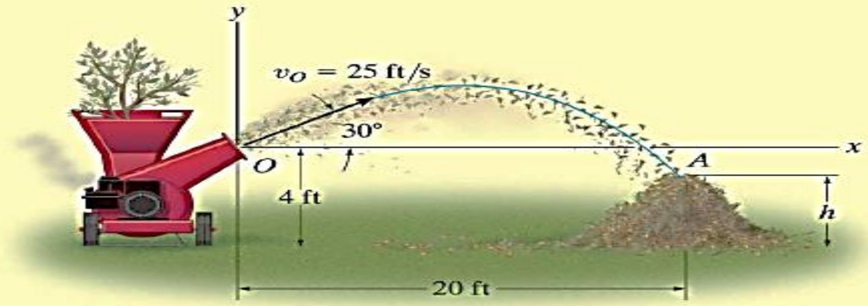


Fig. 12–22

SOLUTION

Coordinate System. When the motion is analyzed between points O and A, the three unknowns are the height h, time of flight t_{OA}, and vertical component of velocity $(v_A)_y$. [Note that $(v_A)_x = (v_O)_x$.] With the origin of coordinates at O, Fig. 12–22, the initial velocity of a chip has components of

$$(v_O)_x = (25 \cos 30°)\text{ ft/s} = 21.65 \text{ ft/s} \rightarrow$$
$$(v_O)_y = (25 \sin 30°)\text{ ft/s} = 12.5 \text{ ft/s} \uparrow$$

Also, $(v_A)_x = (v_O)_x = 21.65$ ft/s and $a_y = -32.2 \text{ ft/s}^2$. Since we do not need to determine $(v_A)_y$, we have

Horizontal Motion.

$$(\overset{+}{\rightarrow}) \qquad x_A = x_O + (v_O)_x t_{OA}$$
$$20 \text{ ft} = 0 + (21.65 \text{ ft/s}) t_{OA}$$
$$t_{OA} = 0.9238 \text{ s}$$

Vertical Motion. Relating t_{OA} to the initial and final elevations of a chip, we have

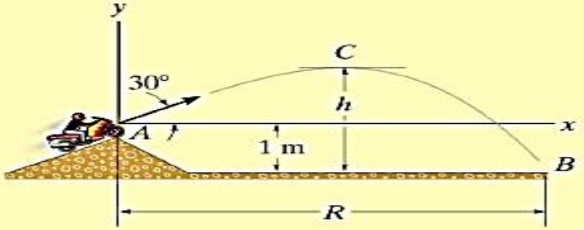
$$(+\uparrow) \qquad y_A = y_O + (v_O)_y t_{OA} + \tfrac{1}{2} a_c t_{OA}^2$$
$$(h - 4 \text{ ft}) = 0 + (12.5 \text{ ft/s})(0.9238 \text{ s}) + \tfrac{1}{2}(-32.2 \text{ ft/s}^2)(0.9238 \text{ s})^2$$
$$h = 1.81 \text{ ft} \qquad \textit{Ans.}$$

NOTE: We can determine $(v_A)_y$ by using $(v_A)_y = (v_O)_y + a_c t_{OA}$.

EXAMPLE 12.13

The track for this racing event was designed so that riders jump off the slope at 30°, from a height of 1 m. During a race it was observed that the rider shown in Fig. 12–23*a* remained in mid air for 1.5 s. Determine the speed at which he was traveling off the ramp, the horizontal distance he travels before striking the ground, and the maximum height he attains. Neglect the size of the bike and rider.

(a)



(b)

Fig. 12–23

SOLUTION

Coordinate System. As shown in Fig. 12–23*b*, the origin of the coordinates is established at *A*. Between the end points of the path *AB* the three unknowns are the initial speed v_A, range R, and the vertical component of velocity $(v_B)_y$.

Vertical Motion. Since the time of flight and the vertical distance between the ends of the path are known, we can determine v_A.

$(+\uparrow)$ $$y_B = y_A + (v_A)_y t_{AB} + \tfrac{1}{2} a_c t_{AB}^2$$

$$-1\text{ m} = 0 + v_A \sin 30°(1.5\text{ s}) + \tfrac{1}{2}(-9.81\text{ m/s}^2)(1.5\text{ s})^2$$

$$v_A = 13.38\text{ m/s} = 13.4\text{ m/s} \qquad \textit{Ans.}$$

Horizontal Motion. The range R can now be determined.

$(\xrightarrow{+})$ $$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 13.38 \cos 30°\text{ m/s}(1.5\text{ s})$$

$$= 17.4\text{ m} \qquad \textit{Ans.}$$

In order to find the maximum height h we will consider the path *AC*, Fig. 12–23*b*. Here the three unknowns are the time of flight t_{AC}, the horizontal distance from *A* to *C*, and the height h. At the maximum height $(v_C)_y = 0$, and since v_A is known, we can determine h *directly* without considering t_{AC} using the following equation.

$$(v_C)_y^2 = (v_A)_y^2 + 2a_c[y_C - y_A]$$

$$0^2 = (13.38 \sin 30°\text{ m/s})^2 + 2(-9.81\text{ m/s}^2)[(h - 1\text{ m}) - 0]$$

$$h = 3.28\text{ m} \qquad \textit{Ans.}$$

NOTE: Show that the bike will strike the ground at *B* with a velocity having components of

$$(v_B)_x = 11.6\text{ m/s} \rightarrow, \quad (v_B)_y = 8.02\text{ m/s}\downarrow$$

Rotation

❑ Planar Rigid-Body Motion.

- The study of planar kinematics of a rigid body is important for the design of gears, cams, and mechanisms used for many mechanical operations.
- Once the kinematics is thoroughly understood, then we can apply the equations of motion, which relate the forces on the body to the body's motion.
- The *planar motion* of a body occurs when all the particles of a rigid body move along paths which are equidistant from a fixed plane.
- There are three types of rigid body planar motion, in order of increasing complexity, they are

❖ Translation.

- This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion.
- When the paths of motion for any two points on the body are parallel lines, the motion is called *rectilinear translation*, Fig.1a.
- If the paths of motion are along curved lines which are equidistant, the motion is called *curvilinear translation*, Fig.1b.

❖ Rotation about a fixed axis.

- When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along circular paths, Fig. 1c.

❖ General plane motion.

- When a body is subjected to general plane motion, it undergoes a combination of translation and rotation, Fig. 1d.
- The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.

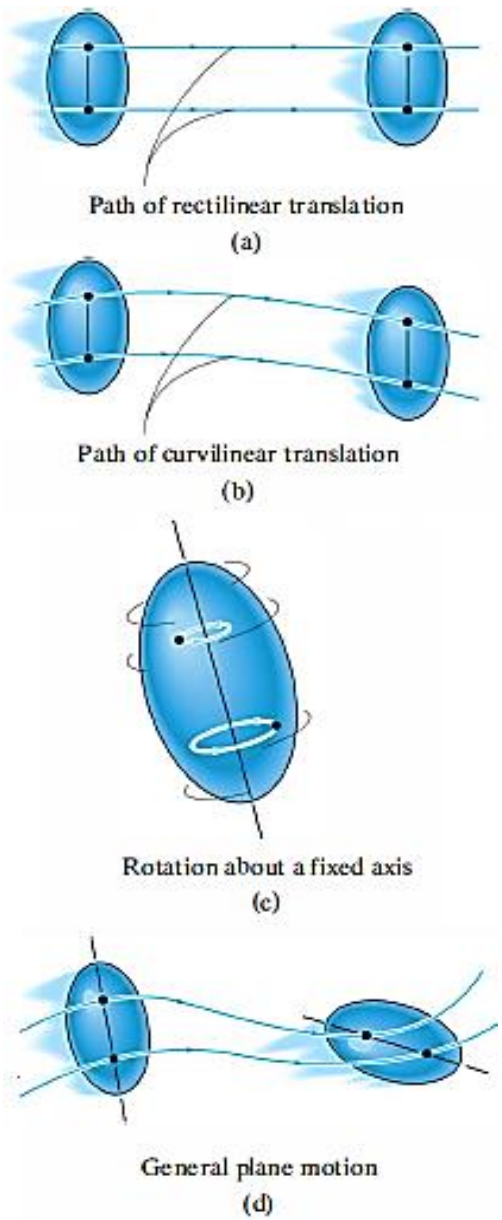


Fig.1.

❖ Rotation a bout a Fixed Axis

- When a body rotates about a fixed axis, any point P located in the body travels along a *circular path*.
- To study this motion it is first necessary to discuss the *angular motion* of the body about the axis.

❑ Angular Motion.

- Since a point is without dimension, it cannot have angular motion. Only lines or bodies undergo angular motion.
- For example, consider the body shown in Fig.2a and the angular motion of a radial line r located within the shaded plane.

❑ Angular Position.

- At the instant shown, the *angular position* of r is defined by the angle θ, measured from a $fixed$ reference line to r.

❑ Angular Displacement.

- The change in the angular position, which can be measured as a differential $d\theta$, is called the angular displacement.
- This vector has a magnitude of $d\theta$, measured in degrees, radians, or revolutions, where $1\,rev = 2\pi\,rad$.
- Since motion is about $a\ fixed\ axis$, the direction of $d\theta$ is $always$ along this axis.
- The direction is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or $d\theta$, points upward, Fig. 2a.
- In two dimensions, as shown by the top view of the shaded plane, Fig.2b, both θ and $d\theta$ are counterclockwise, and so the thumb points outward from the page.

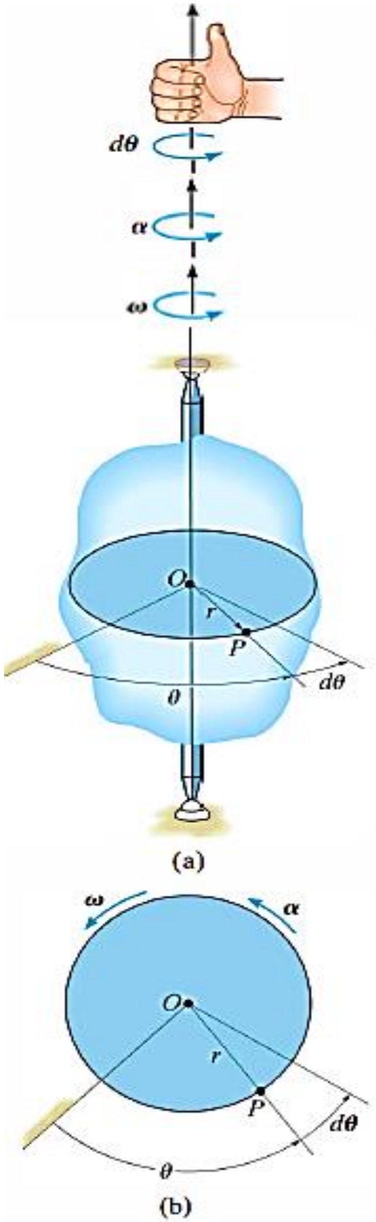


Fig.2.

❑ Angular Velocity.

- The time rate of change in the angular position is called the *angular velocity* ω (omega). Since $d\theta$ occurs during an instant of time dt, then,

$$(\circlearrowleft +) \qquad \boxed{\omega = \frac{d\theta}{dt}} \quad \ldots\ldots\ldots(1)$$

- This vector has a magnitude which is often measured in *rad/s.*
- It is expressed here in scalar form since its *direction* is also along the axis of rotation, Fig.2a.
- When indicating the angular motion in the shaded plane, Fig.2b, we can refer to the sense of rotation as clockwise or counterclockwise.
- Here we have *arbitrarily* chosen counterclockwise rotations as *positive* and indicated this by the curl shown in parentheses next to Eq.1.
- Realize, however, that the directional sense of ω is actually outward from the page.

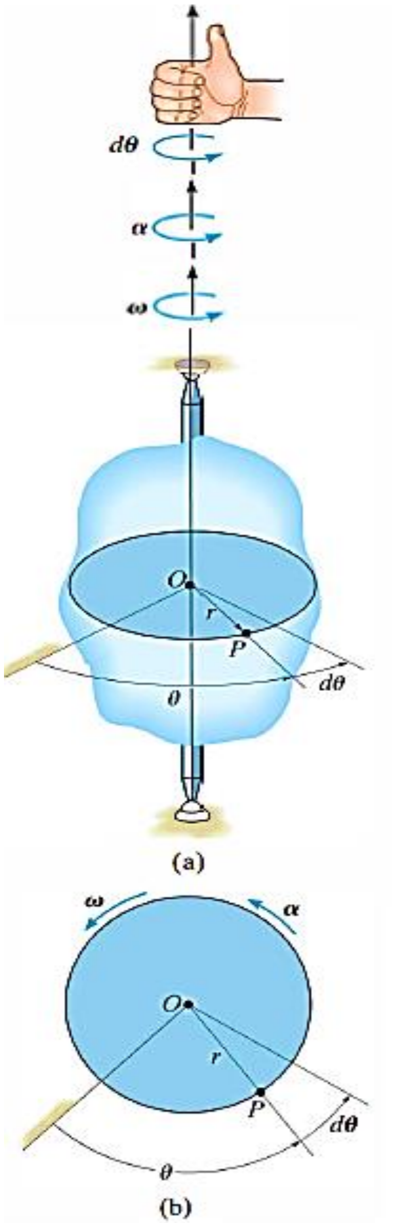


Fig.2.

❖ Angular Acceleration.

- The angular acceleration α **(alpha)** measures the *time rate of change of the angular velocity.*

- The *magnitude* of this vector is

$$(\circlearrowleft +) \qquad \boxed{\alpha = \frac{d\omega}{dt}} \qquad \ldots\ldots(2)$$

- By Eq. 1, it is also possible to express α as

$$(\circlearrowleft +) \qquad \alpha = \frac{d^2\theta}{dt^2} \qquad \ldots\ldots(3)$$

- The line of action of α is the same as that for ω, Fig.2a; however, its sense of direction depends on whether ω is increasing or decreasing.

- If ω is decreasing, then α is called an angular deceleration and therefore has a sense of direction which is opposite to ω .

- By eliminating dt from Eqs.1 and 2, we obtain a differential relation between the angular acceleration, angular velocity, and angular displacement, namely,

$$(\circlearrowleft +) \qquad \boxed{\alpha\, d\theta = \omega\, d\omega} \qquad \ldots\ldots(4)$$

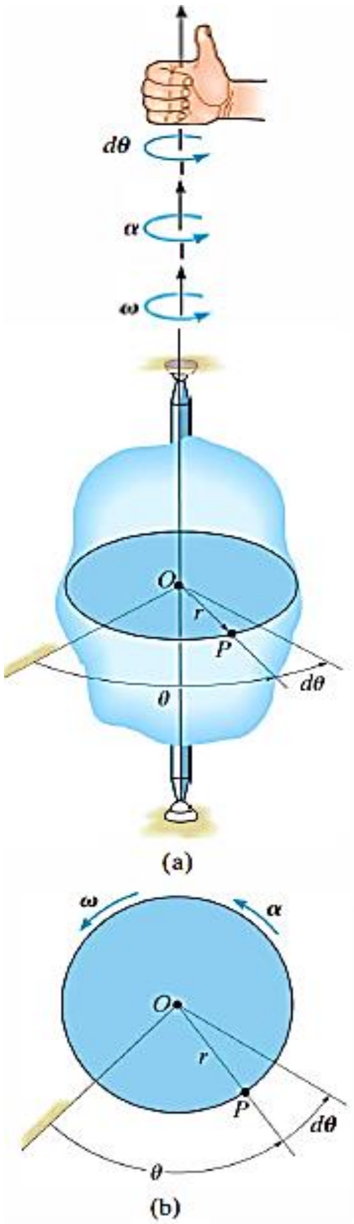


Fig.2.

❑ Constant Angular Acceleration..

- If the angular acceleration of the body is **constant**, $\alpha = \alpha_C$, then $Eqs. 1, 2, and\ 4$, when integrated, yield a set of formulas which relate the body's angular velocity, angular position, and time. These equations are similar to used for rectilinear motion. The results are

$$(\circlearrowleft +) \qquad \omega = \omega_0 + \alpha_c t \qquad \cdots \cdots (5)$$

$$(\circlearrowleft +) \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2 \qquad \cdots \cdots (6)$$

$$(\circlearrowleft +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0) \qquad \cdots \cdots (7)$$

Constant Angular Acceleration

- Where:

$\boldsymbol{\theta_o}$ and $\boldsymbol{\omega_o}$ are the initial values of the body's *angular position* and *angular velocity*, respectively.

Motion of Point P.

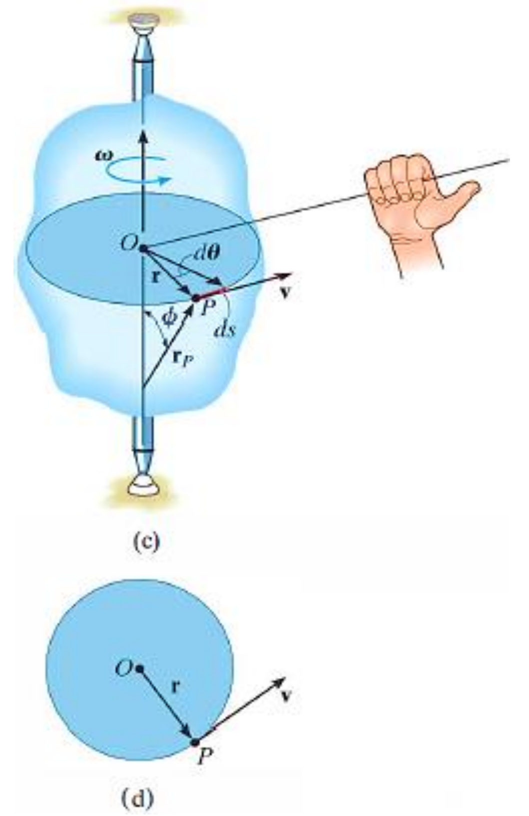


Fig.2.

- As the rigid body in Fig. 2c rotates, point P travels along a circular path of radius r with center at point O. This path is contained within the shaded plane shown in top view, Fig. 2d .

❖ Position and Displacement.

- The position of P is defined by the position vector $\boldsymbol{r}$, which extends from $\boldsymbol{O}$ to **P**.
- If the body rotates $\boldsymbol{d\theta}$ then **P** will displace $ds = r\,d\theta.$

❖ Velocity.

- The velocity of P has a magnitude which can be found by dividing $ds = r\,d\theta$ by dt so that

$$\boxed{v = \omega r} \quad \ldots\ldots(8)$$

- As shown in Figs. 2c and 2d, the direction of **v** is tangent to the circular path.

- Both the magnitude and direction of **v** can also be accounted for by using the cross product of ω and $r_{p.}$
- Where $r_{p.}$ is directed from any point on the axis of rotation to point **P**, Fig. 2c. We have

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P} \quad \ldots\ldots(9)$$

- The order of the vectors in this formulation is important, since the cross product is not commutative, i.e., $\omega \times r_p \neq r_p \times \omega$
- **r** lies in the plane of motion and again the velocity of point P is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad \ldots\ldots(10)$$

❖ Acceleration.

- The acceleration of P can be expressed in terms of its normal and tangential components.
- Since $a_t = dv/dt$ and $a_n = v^2/\rho$, where $\rho = r, v = \omega r$, and $\alpha = d\omega/dt$, we have

$$\boxed{a_t = \alpha r} \quad \cdots\cdots(11)$$

$$\boxed{a_n = \omega^2 r} \quad \cdots\cdots(12)$$

- The *tangential component of acceleration*, Figs. 2e and 2f, represents the time rate of change in the velocity's magnitude.
- If the speed of P is increasing, then $\boldsymbol{a_t}$ acts in the same direction as ***v.***
- If the speed is decreasing, $\boldsymbol{a_t}$ acts in the opposite direction of ***v.***
- If the speed is constant, $\boldsymbol{a_t}$ is zero.
- The *normal component of acceleration* represents the time rate of change in the velocity's direction.
- The direction of $\boldsymbol{a_n}$ is always toward O, the center of the circular path, Figs. 2e and 2f.
- The acceleration of point P can be expressed in terms of the vector cross product.
- Taking the time derivative of Eq. 9 we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt,$$

$$d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$$

yields

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) \cdots\cdots (13)$$

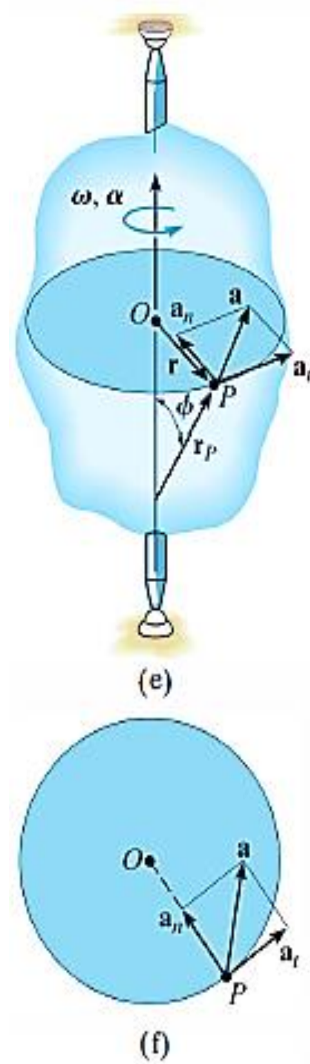


Fig.2.

- Eq. 13 can be identified by its two components as

$$\boxed{\begin{aligned}\mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}\end{aligned}} \quad \cdots\cdots (14)$$

- Since a_t and a_n are perpendicular to one another, if needed the magnitude of acceleration can be determined from the Pythagorean theorem; namely $a = \sqrt{a_n^2 + a_t^2}$, Fig.2 *f*.

Procedure for Analysis

The velocity and acceleration of a point located on a rigid body that is rotating about a fixed axis can be determined using the following procedure.

Angular Motion.

- Establish the positive sense of rotation about the axis of rotation and show it alongside each kinematic equation as it is applied.
- If a relation is known between any *two* of the four variables α, ω, θ, and t, then a third variable can be obtained by using one of the following kinematic equations which relates all three variables.

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha\, d\theta = \omega\, d\omega$$

- If the body's angular acceleration is *constant*, then the following equations can be used:

$$\omega = \omega_0 + \alpha_c t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

- Once the solution is obtained, the sense of θ, ω, and α is determined from the algebraic signs of their numerical quantities.

Motion of Point *P*.

- In most cases the velocity of P and its two components of acceleration can be determined from the scalar equations

$$v = \omega r$$
$$a_t = \alpha r$$
$$a_n = \omega^2 r$$

- If the geometry of the problem is difficult to visualize, the following vector equations should be used:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}_P = \boldsymbol{\alpha} \times \mathbf{r}$$
$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) = -\omega^2 \mathbf{r}$$

- Here $\mathbf{r}_P$ is directed from any point on the axis of rotation to point P, whereas $\mathbf{r}$ lies in the plane of motion of P. Either of these vectors, along with $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, should be expressed in terms of its $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components, and, if necessary, the cross products determined using a determinant expansion (see Eq. B–12).

EXAMPLE 1

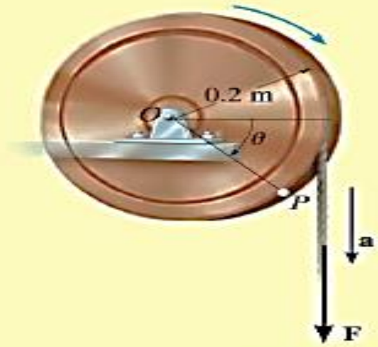


Fig. 5

A cord is wrapped around a wheel in Fig. 5, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \mathrm{m/s^2}$, where t is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line OP in radians.

SOLUTION

Part (a). The wheel is subjected to rotation about a fixed axis passing through point O. Thus, point P on the wheel has motion about a circular path, and the acceleration of this point has *both* tangential and normal components. The tangential component is $(a_P)_t = (4t)\ \mathrm{m/s^2}$, since the cord is wrapped around the wheel and moves *tangent* to it. Hence the angular acceleration of the wheel is

$$(\circlearrowright +) \qquad (a_P)_t = \alpha r$$

$$(4t)\ \mathrm{m/s^2} = \alpha(0.2\ \mathrm{m})$$

$$\alpha = (20t)\ \mathrm{rad/s^2} \circlearrowright$$

Using this result, the wheel's angular velocity ω can now be determined from $\alpha = d\omega/dt$, since this equation relates α, t, and ω. Integrating, with the initial condition that $\omega = 0$ when $t = 0$, yields

$$(\circlearrowright +) \qquad \alpha = \frac{d\omega}{dt} = (20t)\ \mathrm{rad/s^2}$$

$$\int_0^\omega d\omega = \int_0^t 20t\ dt$$

$$\omega = 10t^2\ \mathrm{rad/s} \circlearrowright$$

Part (b). Using this result, the angular position θ of OP can be found from $\omega = d\theta/dt$, since this equation relates θ, ω, and t. Integrating, with the initial condition $\theta = 0$ when $t = 0$, we have

$$(\circlearrowright +) \qquad \frac{d\theta}{dt} = \omega = (10t^2)\ \mathrm{rad/s}$$

$$\int_0^\theta d\theta = \int_0^t 10t^2\ dt$$

$$\theta = 3.33t^3\ \mathrm{rad}$$

NOTE: We cannot use the equation of constant angular acceleration, since α is a function of time.

EXAMPLE 16.2

The motor shown in the photo is used to turn a wheel and attached blower contained within the housing. The details of the design are shown in Fig. 16–6*a*. If the pulley A connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$, determine the magnitudes of the velocity and acceleration of point P on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.

SOLUTION

Angular Motion. First we will convert the two revolutions to radians. Since there are 2π rad in one revolution, then

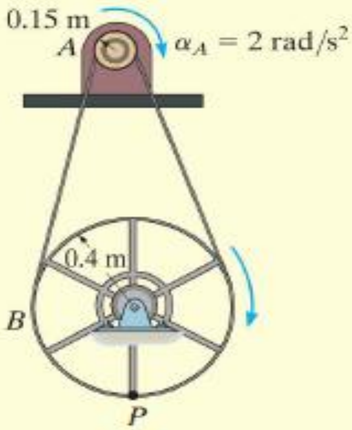
$$\theta_A = 2\ \text{rev}\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 12.57\ \text{rad}$$

Since α_A is constant, the angular velocity of pulley A is therefore

$$(\circlearrowright +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$\omega_A^2 = 0 + 2(2\ \text{rad/s}^2)(12.57\ \text{rad} - 0)$$

$$\omega_A = 7.090\ \text{rad/s}$$



(a)

The belt has the same speed and tangential component of acceleration as it passes over the pulley and wheel. Thus,

$$v = \omega_A r_A = \omega_B r_B;\ \ 7.090\ \text{rad/s}\ (0.15\ \text{m}) = \omega_B(0.4\ \text{m})$$

$$\omega_B = 2.659\ \text{rad/s}$$

$$a_t = \alpha_A r_A = \alpha_B r_B;\ \ 2\ \text{rad/s}^2\ (0.15\ \text{m}) = \alpha_B(0.4\ \text{m})$$

$$\alpha_B = 0.750\ \text{rad/s}^2$$

Motion of *P*. As shown on the kinematic diagram in Fig. 16–6*b*, we have

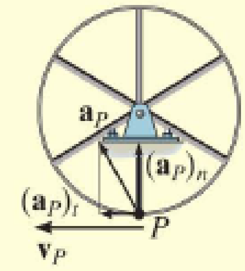
$$v_P = \omega_B r_B = 2.659 \text{ rad/s } (0.4 \text{ m}) = 1.06 \text{ m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha_B r_B = 0.750 \text{ rad/s}^2 (0.4 \text{ m}) = 0.3 \text{ m/s}^2$$

$$(a_P)_n = \omega_B^2 r_B = (2.659 \text{ rad/s})^2 (0.4 \text{ m}) = 2.827 \text{ m/s}^2$$

Thus

$$a_P = \sqrt{(0.3 \text{ m/s}^2)^2 + (2.827 \text{ m/s}^2)^2} = 2.84 \text{ m/s}^2 \qquad \textit{Ans.}$$



(b)

Fig. 16–6

Kinetics of a Particle :Work and Energy

❑ The Work of a Force.

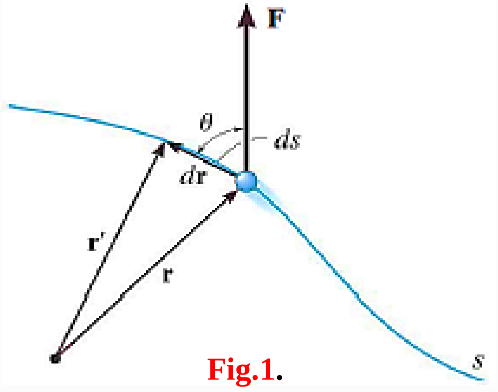


Fig.1.

- We will analyze motion of a particle using the concepts of work and energy.
- The resulting equation will be useful for solving problems that involve force, velocity, and displacement.
- A force F will do work on a particle only when the particle undergoes a *displacement in the direction of the force.*
- If the force **F** in Fig.1 causes the particle to move along the path $\boldsymbol{s}$ from position $\boldsymbol{r}$ to a new position $\boldsymbol{r'}$, the displacement is then $\boldsymbol{dr = r' - r}$.
- The magnitude of $\boldsymbol{dr}$ is $\boldsymbol{ds}$, the length of the differential segment along the path..
- If the angle between the tails of d**r** and **F** is $\boldsymbol{\theta}$, Fig.1 , then the work done by **F** is a *scalar quantity*, defined by

$$dU = F\,ds\cos\theta$$

- This equation can also be written as

$$dU = F.dr$$

- This result may be interpreted in one of two ways:

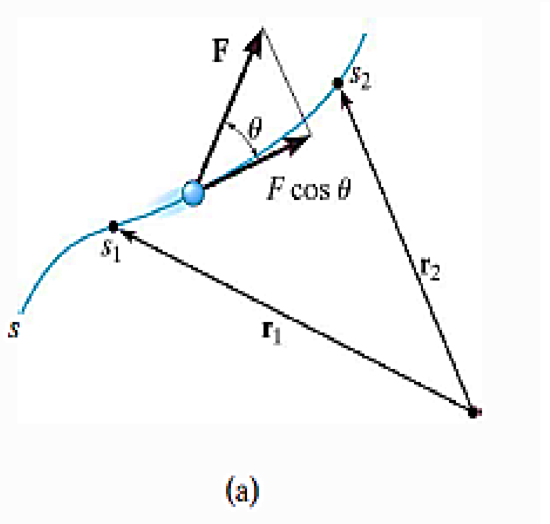
▪ As the product of $\boldsymbol{F}$ and the component of displacement $\boldsymbol{ds\cos\theta}$ in the direction of the force.

▪ Or , as the product of $\boldsymbol{ds}$ and the component of force, $\boldsymbol{F\cos\theta}$, in the direction of displacement.

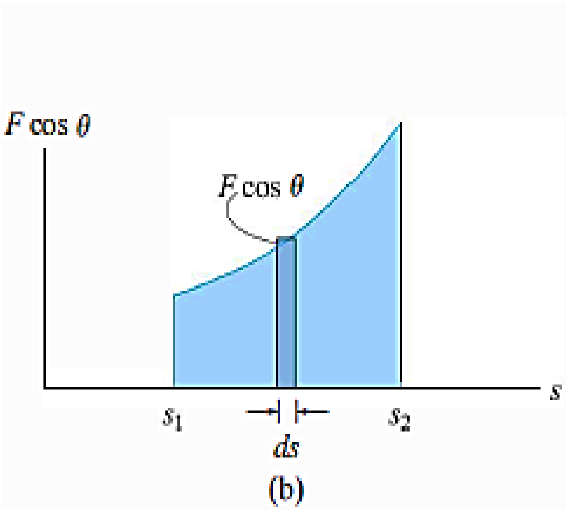
- If $0° \leq \theta < 90°$, then the force component and the displacement have the same sense so that the $work\ is\ positive$.
- If $90° < \theta \leq 180°$, these vectors will have opposite sense, and therefore the $work\ is\ negative$.
- $dU = 0$ if the force is $perpendicular$ to displacement, since $\cos 90° = 0$.
- If the force is applied at a $fixed\ point$, in which case the $displacement\ is\ zero$.
- The unit of work in SI units is the $joule\ (J)$, which is the amount of work done by a $one - newton\ force$ when it moves through a distance of $one\ meter$ in the direction of the force $(1\,J = 1\,N \cdot m)$.
- In the $foot\text{–}pound\text{–}second\ (FPS)\ system$, work is measured in units of foot-pounds $(\boldsymbol{ft\,.lb})$, which is the work done by a one-pound force acting through a distance of one foot in the direction of the force.

❖ Work of a Variable Force.

- If the particle acted upon by the force F undergoes a finite displacement along its path from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ or $\boldsymbol{s_1}$ to $\boldsymbol{s_2}$, Fig.2a, the work of force **F** is determined by integration. Provided **F** and θ can be expressed as a function of position, then.



$$U_{1-2} = \int_{\boldsymbol{r_1}}^{\boldsymbol{r_2}} \boldsymbol{F}.d\mathbf{r} = \int_{s_1}^{s_2} F\cos\theta\, ds \quad \text{........(1)}$$



- This relation may be obtained by using experimental data to plot a graph of $F\cos\theta$ vs. s.
- Then the area under this graph bounded by s_1 and s_2 represents the total work, Fig.2b.

Fig.2.

❖ Work of a Constant Force Moving Along a Straight Line.

- If the force $\boldsymbol{F_c}$ has a constant magnitude and acts at a constant angle θ from its straight-line path, Fig. 3a, then the component of $\boldsymbol{F_c}$ in the direction of displacement is always $\boldsymbol{F_c}\cos\theta$.
- The work done by $\boldsymbol{F_c}$ when the particle is displaced from s_1 to s_2 is determined from Eq. 1, in which case

$$U_{1-2} = \boldsymbol{F_c}\cos\theta\int_{s_1}^{s_2} ds \quad or \quad U_{1-2} = \boldsymbol{F_c}\cos\theta\,(s_2 - s_1) \;\ldots\ldots\ldots(2)$$

- Here the work of $\boldsymbol{F_c}$ represents the area of the rectangle in Fig.3b.

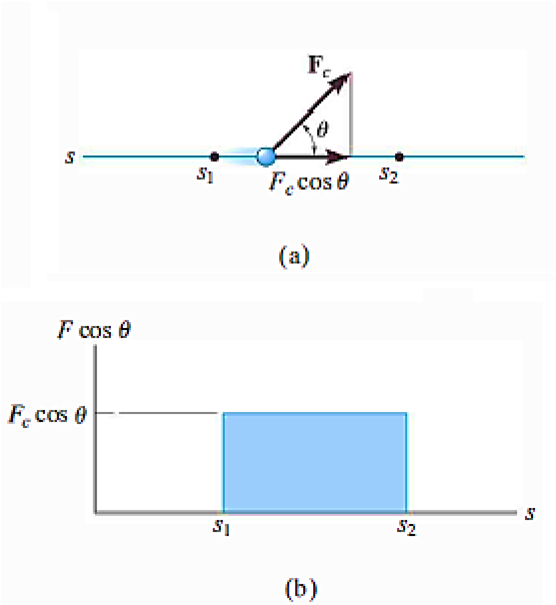


Fig.3.

❖ Work of a Weight.

- Consider a particle of weight **W**, which moves up along the path s shown in Fig.4 from position s_1 to position s_2.
- At an intermediate point, the displacement $d\mathbf{r} = dx_i + dy_j + dz_k$.
- Since $\boldsymbol{W} = -\boldsymbol{W_j}$, applying Eq.1. we have

$$U_{1-2} = \int_{r_1}^{r_2} \boldsymbol{F}.\,d\mathbf{r} = \int_{r_1}^{r_2} (-\boldsymbol{W_j})(\,dx_i + dy_j + dz_k)$$

$$= \int_{y_1}^{y_2} -\boldsymbol{W}dy = -\boldsymbol{W}(\boldsymbol{y_2} - y_1)$$

$$U_{1-2} = -\mathbf{W}\Delta y \qquad \ldots\ldots\ldots(3)$$

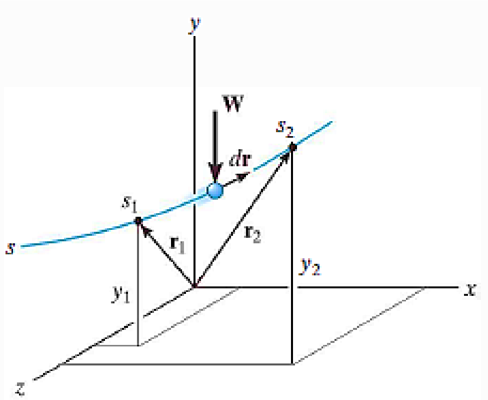


Fig.4.

- The work is independent of the path and is equal to the magnitude of the particle's weight times its vertical displacement.

➢ In the case shown in Fig. 4 the work is negative, since $\boldsymbol{W}$ is downward and $\Delta\boldsymbol{y}$ is upward.

➢ If the particle is displaced downward $(-\Delta y)$, the work of the weight is positive.

❖ Work of a Spring Force.

- If an elastic spring is elongated a distance ds, Fig. 5a, then the work done by the force that acts on the attached particle is

$$dU = -F_s ds = -ksds.$$

- The work is negative since $\boldsymbol{F_s}$ acts in the opposite sense to ds.

- If the particle displaces from s_1 to s_2, the work of $\boldsymbol{F_s}$ is then

$$U_{1-2} = \int_{s_1}^{s_2} F_s ds = \int_{s_1}^{s_2} -ksds$$

$$U_{1-2} = -(\frac{1}{2}k\, s_2{}^2 - \frac{1}{2}k\, s_1{}^2) \qquad \text{........(4)}$$

- This work represents the trapezoidal area under the line $\boldsymbol{F_s = ks}$, Fig. 5b.

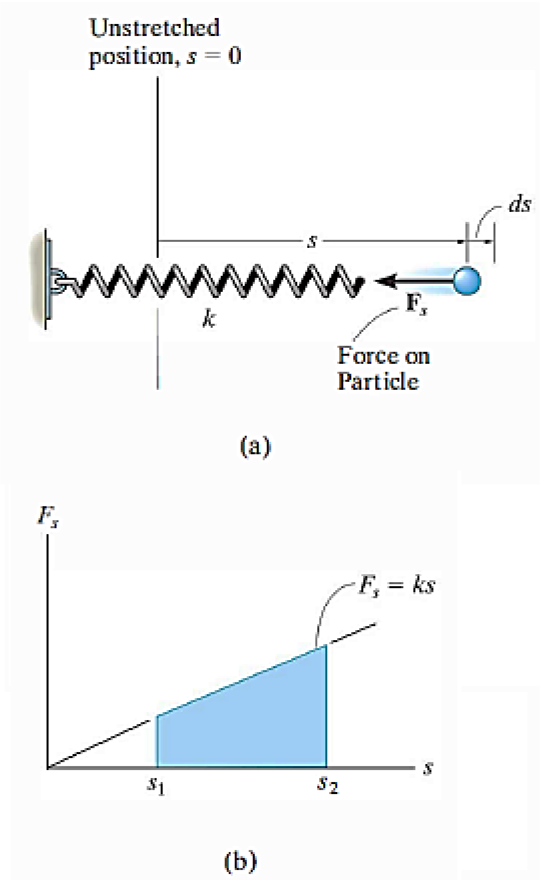


Fig.5.

❑ Principle of Work and Energy.

- Consider the particle in Fig.6, which is located on the path defined relative to an inertial coordinate system.

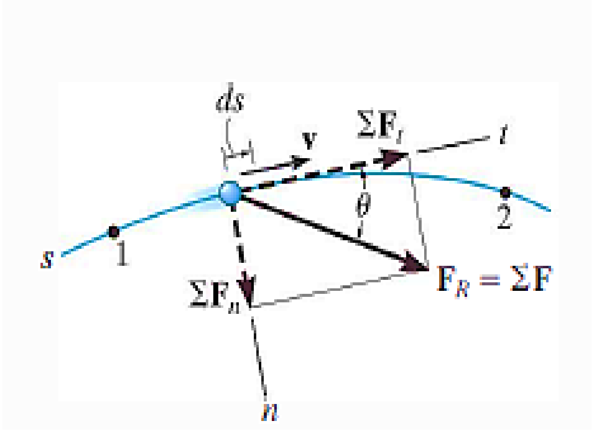


Fig.6.

- If the particle has a **mass m** and is subjected to a system of external forces represented by the resultant $\boldsymbol{F}_R = \sum \boldsymbol{F}$, then the equation of motion for the particle in the tangential direction is $\sum \boldsymbol{F}_t = ma_t$.

- Applying the kinematic equation $a_t = v\, dv/ds$ and integrating both sides, assuming initially that the particle has a position $s = s_1$ and a speed $v = v_1$, and later at $s = s_2, v = v_2$, we have

$$\Sigma \int_{s_1}^{s_2} F_t\, ds = \int_{v_1}^{v_2} mv\, dv$$

$$\Sigma \int_{s_1}^{s_2} F_t\, ds = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad \text{........(5)}$$

- From Fig.6, note that $\sum F_t = \sum F\cos\theta$, and since work is defined from Eq.1, the final result can be written as

$$\sum U_{1-2} = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2 \qquad \text{........(6)}$$

- This equation represents the principle of work and energy for the particle.
- The term on the left is the sum of the work done by all the forces acting on the particle as the particle moves from point 1 to point 2.
- The terms $T = \frac{1}{2}mv^2$ define the particle's final and initial kinetic energy.
- Work, kinetic energy is a scalar and has units of joules (J). Work, which can be either positive or negative but the kinetic energy is always positive, regardless of the direction of motion of the particle.

- Eq.6 is applied, it is often expressed in the form

$$T_1 + \sum U_{1-2} = T_2 \quad \text{........(7)}$$

- Eq. 7. states that the particle's initial kinetic energy plus the work done by all the forces acting on the particle as it moves from its initial to its final position is equal to the particle's final kinetic energy.

Principle of Work and Energy for a System of Particles.

- The principle of work and energy can be extended to include a system of particles isolated within an enclosed region of space as shown in **Fig. 7**.
- The arbitrary ith particle, having a **mass** $\boldsymbol{m_i}$, is subjected to a resultant external force $\boldsymbol{F_i}$ and a resultant internal force $\boldsymbol{f_i}$ which all the other particles exert on the ith particle.
- If we apply the principle of work and energy to this and each of the other particles in the system, then since work and energy are scalar quantities, the equations can be summed algebraically, which gives

$$\sum T_1 + \sum U_{1-2} = \sum T_2 \quad \ldots\ldots..(8)$$

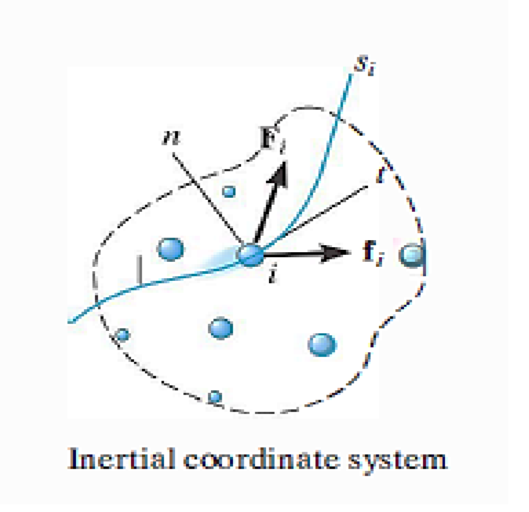


Fig. 7.

- In this case, the initial kinetic energy of the system plus the work done by all the external and internal forces acting on the system is equal to the final kinetic energy of the system.

❖ Work of Friction Caused by Sliding.

- A special class of problems will now be investigated which requires a careful application of Eq. 8.

- These problems involve cases where a body slides over the surface of another body in the presence of friction.
- For example, a block which is translating a distance s over a rough surface as shown in Fig. 8a.
- If the applied force $\boldsymbol{P}$ just balances the *resultant* frictional force $\mu_k N$, Fig.8b, then due to equilibrium a constant velocity **v** is maintained, and one would expect Eq. 8. to be applied as follows:

$$\frac{1}{2}mv^2 + Ps - \mu_k Ns = \frac{1}{2}mv^2$$

- Indeed this equation is satisfied if $\boldsymbol{P} = \boldsymbol{\mu_k N}$; however, as one realizes from experience, the sliding motion will generate heat, a form of energy which seems not to be accounted for in the work-energy equation.

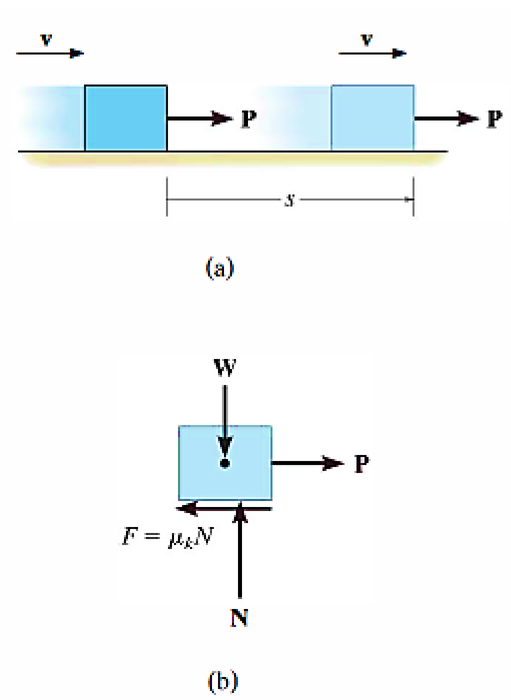
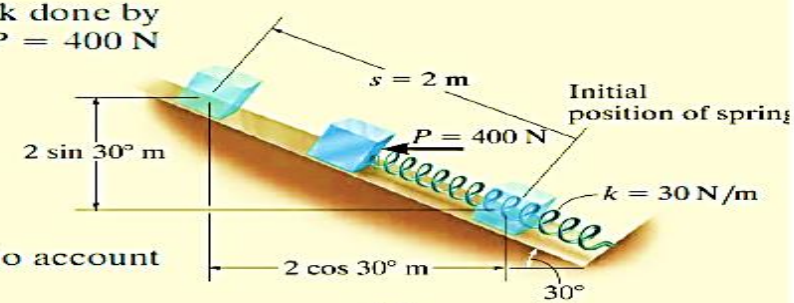


Fig.8.

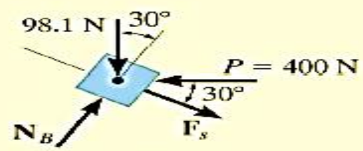
- In summary then, Eq. 8 can be applied to problems involving sliding friction; however, it should be fully realized that the work of the resultant frictional force is not represented by $\boldsymbol{\mu_k N s}$; instead, this term represents *both* the external work of friction $(\boldsymbol{\mu_k N s'})$ and internal work $[\boldsymbol{\mu_k N}\ (\boldsymbol{s} - \boldsymbol{s'})]$ which is converted into various forms of internal energy, such as heat.

EXAMPLE 1

The 10-kg block shown in Fig. 9*a* rests on the smooth incline. If the spring is originally stretched 0.5 m, determine the total work done by all the forces acting on the block when a horizontal force $P = 400$ N pushes the block up the plane $s = 2$ m.



(a)



(b)

Fig. 9

SOLUTION

First the free-body diagram of the block is drawn in order to account for all the forces that act on the block, Fig. 9*b*.

Horizontal Force *P*. Since this force is *constant*, the work is determined using Eq. -2. The result can be calculated as the force times the component of displacement in the direction of the force; i.e.,

$$U_P = 400 \text{ N } (2 \text{ m} \cos 30°) = 692.8 \text{ J}$$

or the displacement times the component of force in the direction of displacement, i.e.,

$$U_P = 400 \text{ N} \cos 30°(2 \text{ m}) = 692.8 \text{ J}$$

Spring Force F_s. In the initial position the spring is stretched $s_1 = 0.5$ m and in the final position it is stretched $s_2 = 0.5 \text{ m} + 2 \text{ m} = 2.5$ m. We require the work to be negative since the force and displacement are opposite to each other. The work of $\mathbf{F}_s$ is thus

$$U_s = -\left[\tfrac{1}{2}(30 \text{ N/m})(2.5 \text{ m})^2 - \tfrac{1}{2}(30 \text{ N/m})(0.5 \text{ m})^2\right] = -90 \text{ J}$$

Weight *W*. Since the weight acts in the opposite sense to its vertical displacement, the work is negative; i.e.,

$$U_W = -(98.1 \text{ N})\ (2 \text{ m} \sin 30°) = -98.1 \text{ J}$$

Note that it is also possible to consider the component of weight in the direction of displacement; i.e.,

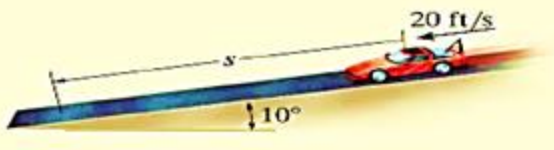
$$U_W = -(98.1 \sin 30° \text{ N})\ (2 \text{ m}) = -98.1 \text{ J}$$

Normal Force N_B. This force does *no work* since it is *always* perpendicular to the displacement.

Total Work. The work of all the forces when the block is displaced 2 m is therefore

$$U_T = 692.8 \text{ J} - 90 \text{ J} - 98.1 \text{ J} = 505 \text{ J} \qquad \textit{Ans.}$$

EXAMPLE .2

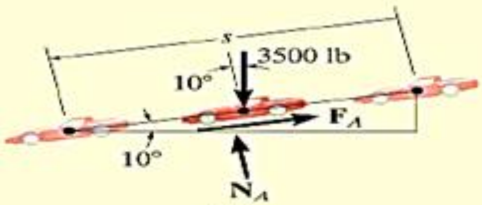


(a)

The 3500-lb automobile shown in Fig. 10*a* travels down the 10° inclined road at a speed of 20 ft/s. If the driver jams on the brakes, causing his wheels to lock, determine how far s the tires skid on the road. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.5$.

SOLUTION

This problem can be solved using the principle of work and energy, since it involves force, velocity, and displacement.



(b)

Fig. 10-

Work (Free-Body Diagram). As shown in Fig. 10*b*, the normal force $\mathbf{N}_A$ does no work since it never undergoes displacement along its line of action. The weight, 3500 lb, is displaced s sin 10° and does positive work. Why? The frictional force $\mathbf{F}_A$ does both external and internal work when it undergoes a displacement s. This work is negative since it is in the opposite sense of direction to the displacement. Applying the equation of equilibrium normal to the road, we have

$$+\nwarrow \Sigma F_n = 0; \quad N_A - 3500 \cos 10^\circ \text{ lb} = 0 \qquad N_A = 3446.8 \text{ lb}$$

Thus,

$$F_A = \mu_k N_A = 0.5\,(3446.8 \text{ lb}) = 1723.4 \text{ lb}$$

Principle of Work and Energy.

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\frac{1}{2}\left(\frac{3500 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(20 \text{ ft/s})^2 + 3500 \text{ lb}(s \sin 10^\circ) - (1723.4 \text{ lb})s = 0$$

Solving for s yields

$$s = 19.5 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If this problem is solved by using the equation of motion, *two steps* are involved. First, from the free-body diagram, Fig. 10*b*, the equation of motion is applied along the incline. This yields

$$+\swarrow \Sigma F_s = ma_s; \quad 3500 \sin 10^\circ \text{ lb} - 1723.4 \text{ lb} = \frac{3500 \text{ lb}}{32.2 \text{ ft/s}^2}a$$

$$a = -10.3 \text{ ft/s}^2$$

Then, since a is constant, we have

$$(+\swarrow) \quad v^2 = v_0^2 + 2a_c(s - s_0);$$

$$(0)^2 = (20 \text{ ft/s})^2 + 2(-10.3 \text{ ft/s}^2)(s - 0)$$

$$s = 19.5 \text{ ft} \qquad \textit{Ans.}$$

Vibrations

- A vibration is the periodic motion of a body or system of connected bodies displaced from a position of equilibrium.
- In general, there are two types of vibration, $free$ and $forced$.
- $Free\ vibration$ occurs when the motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or the vibration of an elastic rod.
- $Forced\ vibration$ is caused by an external periodic or intermittent force applied to the system.
- Both of these types of vibration can either be damped or undamped.
- $Undamped\ vibrations$ can continue indefinitely because frictional effects are neglected in the analysis.
- Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually $damped$.
- The simplest type of vibrating motion is undamped free vibration, represented by the block and spring model shown in Fig. 12 a.

- Vibrating motion occurs when the block is released from a displaced position x so that the spring pulls on the block.
- The block will attain a velocity such that it will proceed to move out of equilibrium when $x = 0$, and provided the supporting surface is smooth, the block will oscillate back and forth.
- The time-dependent path of motion of the block can be determined by applying the equation of motion to the block when it is in the displaced position x.
- The free-body diagram is shown in Fig.12b.
- The elastic restoring force $F = kx$ is always directed toward the equilibrium position, whereas the acceleration a is assumed to act in the direction of *positive displacement*.

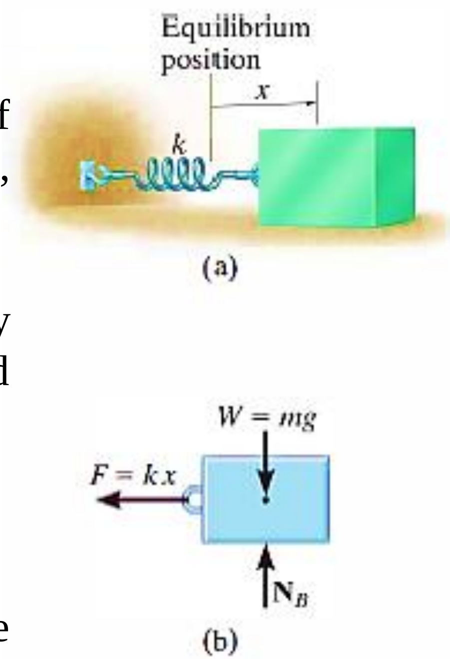


Fig.12.

- Since $a = d^2x/dt^2 = \ddot{x}$, we have → $\xrightarrow{+} \Sigma F_x = ma_x; \quad -kx = m\ddot{x}$
- The acceleration is proportional to the block's displacement
- Motion described in this manner is called *simple harmonic motion*. Rearranging the terms into a "standard form" gives

$$\ddot{x} + \omega_n^2 x = 0 \qquad \text{........(9)}$$

- The constant $\boldsymbol{\omega_n}$ is called the *natural frequency*, and in this case $\boldsymbol{\omega_n} = \sqrt{\dfrac{k}{m}}$ ....(10)

- *Equation 9* can also be obtained by considering the block to be suspended so that the displacement y is measured from the block's equilibrium position, Fig. 13a.
- When the block is in equilibrium, the spring exerts an upward force of $F = W = mg$ on the block.
- when the block is displaced a distance y downward from this position, the magnitude of the spring force is $F = W + ky$, Fig.13b.
- Applying the equation of motion gives

$$+\downarrow \Sigma F_y = ma_y; \qquad -W - ky + W = m\ddot{y}$$

or

$$\ddot{y} + \omega_n^2 y = 0$$

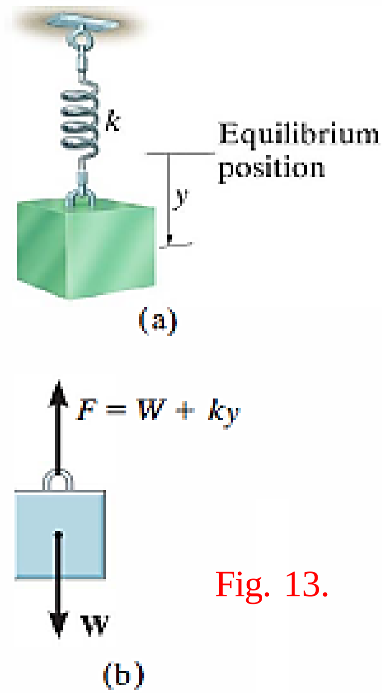


Fig. 13.

- Which is the same form as Eq.9 and $\boldsymbol{\omega_n}$ is defined by Eq.10.
- Equation 9 is a homogeneous, second-order, linear, differential equation with constant coefficients. It can be shown, using the methods of differential equations, that the general solution is

$$x = A \sin \omega_n t + B \cos \omega_n t \quad \ldots\ldots\ldots(11)$$

- Here A and B represent two constants of integration. The block's velocity and acceleration are determined by taking successive time derivatives, which yields

$$v = \dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t \quad \ldots\ldots\ldots(12)$$

$$a = \ddot{x} = -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t \quad \ldots\ldots\ldots(13)$$

- The integration constants in Eq. 11 are generally determined from the initial conditions of the problem.

- For example, suppose that the block in Fig.12a has been displaced a distance x_1 to the right from its equilibrium position and given an initial (positive) velocity v_1 directed to the right.
- Substituting $x = x_1$ when $t = 0$ into Eq. 11 yields $B = x_1$.
- And since $v = v_1$ when $t = 0$, using Eq. 12 we obtain $A = \frac{v_1}{\omega_n}$.
- Substituting these value into Eq. 11, the equation describing the motion becomes

$$x = \frac{v_1}{\omega_n} \sin \omega_n t + x_1 \cos \omega_n t \qquad \text{........(14)}$$

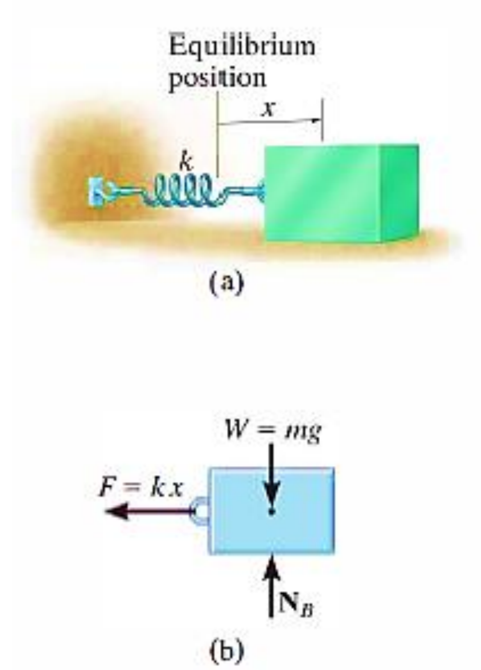


Fig.12.

- Equation 11 may also be expressed in terms of simple sinusoidal motion.
- To show this, let

$$A = C \cos \phi \qquad \text{........(15)}$$

- And

$$B = C \sin \phi \qquad \text{........(16)}$$

- Where C and $\emptyset$ are new constants to be determined in place of A and B. Substituting into Eq. 11 yields

$$x = C \cos \phi \sin \omega_n t + C \sin \phi \cos \omega_n t$$

- And since $\sin(\theta + \emptyset) = \sin\theta \cos\emptyset + \cos\theta \sin\emptyset$, then
-

$$x = C \sin(\omega_n t + \emptyset) \qquad \text{........(17)}$$

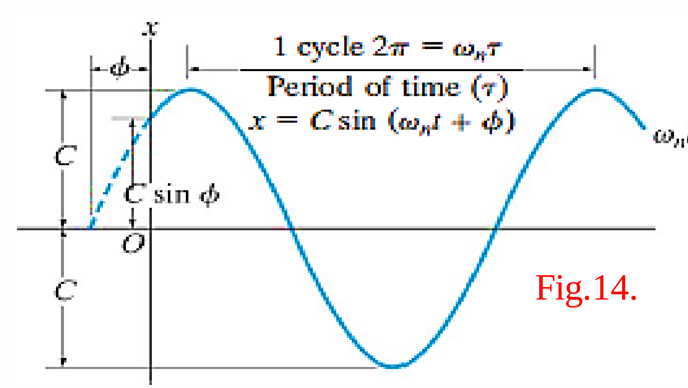


Fig.14.

- If this equation is plotted on an x versus $\omega_n t$ axis, the graph shown in Fig. 14 is obtained.

- The maximum displacement of the block from its equilibrium position is defined as the amplitude of vibration

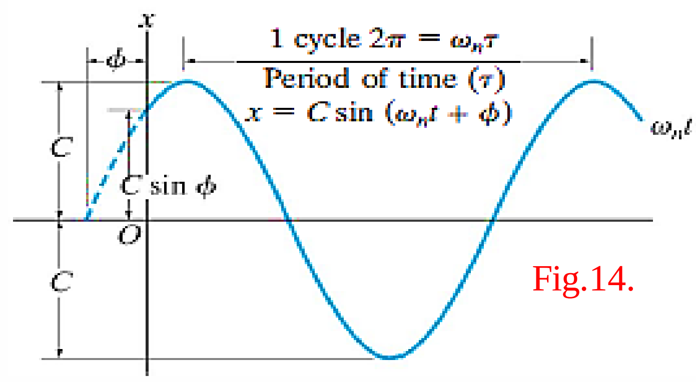


Fig.14.

- From either the Figure 14 or Eq.17 the amplitude is C.

- The angle ∅ is called the *phase angle* since it represents the amount by which the curve is displaced from the origin when $t = 0$.
- We can relate these two constants to A and B using $Eqs. 15\ and\ 16$.

- squaring and adding these two equations, the amplitude becomes $C = \sqrt{A^2 + B^2}$ ……..(18)

- If $Eq. 16$ is divided by Eq. 15 , the phase angle is then $\phi = \tan^{-1}\dfrac{B}{A}$ ……..(19)

- Note that the sine curve, Eq. 17, completes one *cycle* in time $t = \tau\ (tau)$ when $\omega_n\tau = 2\pi$, or

$$\tau = \frac{2\pi}{\omega_n} \quad \text{……..(20)}$$

- This time interval is called a *period*, Fig14. Using Eq.10, the period can also be represented as

$$\tau = 2\pi\sqrt{\frac{m}{k}} \quad \text{……..(21)}$$

- the *frequency* f is defined as the number of cycles completed per unit of time, which is the reciprocal of the period; that is,

$$f = \frac{1}{\tau} = \frac{\omega_n}{2\pi} \quad \text{……..(22)} \qquad \text{or} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{……..(23)}$$

- The frequency is expressed in cycles/s. This ratio of units is called a *hertz* (Hz), where $1\,Hz = 1\dfrac{cycle}{s} = 2\pi\ rad/s$.

EXAMPLE 4

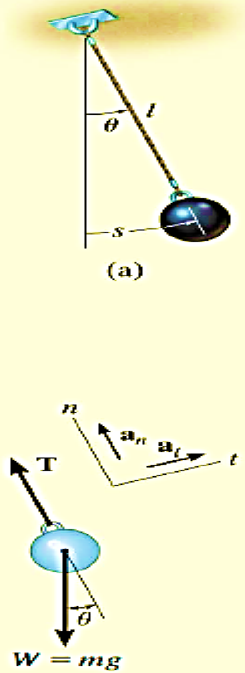


Fig. 15

Determine the period of oscillation for the simple pendulum shown in Fig. 15 *a*. The bob has a mass m and is attached to a cord of length l. Neglect the size of the bob.

SOLUTION

Free-Body Diagram. Motion of the system will be related to the position coordinate $(q =)\ \theta$, Fig. 15 *b*. When the bob is displaced by a small angle θ, the *restoring force* acting on the bob is created by the tangential component of its weight, $mg \sin \theta$. Furthermore, $\mathbf{a}_t$ acts in the direction of *increasing* s (or θ).

Equation of Motion. Applying the equation of motion in the *tangential direction*, since it involves the restoring force, yields

$$+\nearrow \Sigma F_t = ma_t; \qquad -mg \sin \theta = ma_t \tag{1}$$

Kinematics. $a_t = d^2s/dt^2 = \ddot{s}$. Furthermore, s can be related to θ by the equation $s = l\theta$, so that $a_t = l\ddot{\theta}$. Hence, Eq. 1 reduces to

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \tag{2}$$

The solution of this equation involves the use of an elliptic integral. For *small displacements*, however, $\sin \theta \approx \theta$, in which case

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{3}$$

Comparing this equation with Eq. 9 $(\ddot{x} + \omega_n^2 x = 0)$, it is seen that $\omega_n = \sqrt{g/l}$. From Eq. 20, the period of time required for the bob to make one complete swing is therefore

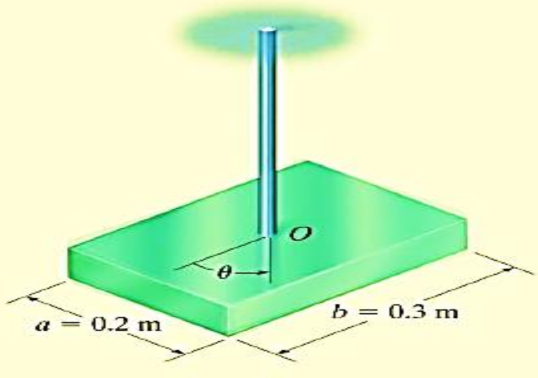
$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}} \qquad \textit{Ans.}$$

This interesting result, originally discovered by Galileo Galilei through experiment, indicates that the period depends only on the length of the cord and not on the mass of the pendulum bob or the angle θ.

NOTE: The solution of Eq. 3 is given by Eq 11, where $\omega_n = \sqrt{g/l}$ and θ is substituted for x. Like the block and spring, the constants A and B in this problem can be determined if, for example, one knows the displacement and velocity of the bob at a given instant.

EXAMPLE 5

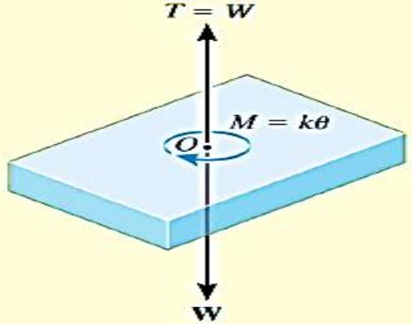
The 10-kg rectangular plate shown in Fig. 16*a* is suspended at its center from a rod having a torsional stiffness $k = 1.5\ \text{N}\cdot\text{m/rad}$. Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



(a)

SOLUTION

Free-Body Diagram. Fig. 16*b*. Since the plate is displaced in its own plane, the torsional *restoring* moment created by the rod is $M = k\theta$. This moment acts in the direction opposite to the angular displacement θ. The angular acceleration $\ddot{\theta}$ acts in the direction of *positive* θ.



(b)

Fig. 16

Equation of Motion.

$$\Sigma M_O = I_O\alpha; \qquad -k\theta = I_O\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{k}{I_O}\theta = 0$$

Since this equation is in the "standard form," the natural frequency is $\omega_n = \sqrt{k/I_O}$.

From the table on the inside back cover, the moment of inertia of the plate about an axis coincident with the rod is $I_O = \frac{1}{12}m(a^2 + b^2)$. Hence,

$$I_O = \frac{1}{12}(10\ \text{kg})[(0.2\ \text{m})^2 + (0.3\ \text{m})^2] = 0.1083\ \text{kg}\cdot\text{m}^2$$

The natural period of vibration is therefore,

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{k}} = 2\pi\sqrt{\frac{0.1083}{1.5}} = 1.69\ \text{s} \qquad \textit{Ans.}$$